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ABSTRACT

This is part one of a two-part manual for teachers using SMSG high school text materials. Each chapter contains a commentary on the text, answers to exercises, and a set of illustrative test questions. Chapter topics include sets, relations and functions, polynomial functions, and algebra of polynomial functions. (MP)

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# MATHEMATICS FOR HIGH SCHOOL
## ELEMENTARY FUNCTIONS (Part 1)
## *COMMENTARY FOR TEACHERS*
(preliminary edition)

SE 028 946

# MATHEMATICS FOR HIGH SCHOOL

## ELEMENTARY FUNCTIONS (Part 1)

*COMMENTARY FOR TEACHERS*

(preliminary edition)

Prepared under the supervision of the Panel on Sample Textbooks of the School Mathematics Study Group:

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## COMMENTARY FOR TEACHERS

## for

# ELEMENTARY FUNCTIONS

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# Commentary for Teachers

# ELEMENTARY FUNCTIONS

## Introduction

The text, Elementary Functions, and this accompanying commentary have been produced by a team of high school and college mathematics teachers for the School Mathematics Study Group for use in the first half of the 12th grade. One obvious consequence of a cooperative writing project of this kind, involving a dozen or more individuals, is that the style of writing and the level of difficulty are not uniform throughout the book. Some chapters are spelled out in greater detail than others, and in general the pace tends to accelerate after the opening chapter. We feel that this unevenness of presentation may be helpful in evaluating the effectiveness of the textbook, since the teachers who use it in their classes during the first year will be expected to report upon the relative success of the various chapters. In the light of these reports, a subsequent revision of the text should produce a more realistic presentation, neither too easy nor too difficult for high school seniors.

The book follows generally the outline recommended by the Commission on Mathematics for the first semester of the 12th grade. (For a copy of the Commission's Report, write to College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey.) At the same time we have not felt bound to observe the Commission's point of view in all respects.

The question of the time required to teach the various topics included in this course has been the subject of much discussion and has resulted in relegating to optional sections or the appendices some material not prerequisite to what follows. At the same time we feel that this material is within the grasp of able students and that it will broaden their mathematical backgrounds appreciably. The optional topics provide a means for making differentiated assignments in a class for which such a procedure would be helpful.

A suggested time schedule for this course is as follows:

| | |
|---|---|
| Chapter 1 | 3 weeks |
| Chapter 2 | 3 weeks |
| Chapter 3 | 3 weeks |
| Chapter 4 | 4 weeks |
| Chapter 5 | 5 weeks |
| | 18 weeks |

It is important to note that Chapter 5, Circular Functions, assumes knowledge of the usual course in analytic trigonometry based upon angles. We do not believe that Chapter 5 is the place to begin the study of trigonometry. Hence, teachers using this textbook with students who have not had such a background are advised to include in the 12th year course a unit on analytic trigonometry at some time before taking up Chapter 5. This might be done at the beginning of the year or it might be sandwiched between Chapters 4 and 5. In any event it would mean that the entire course would require more than half a year to complete.

At the end of each chapter a miscellaneous set of exercises based on the chapter has been included. Since many of the excercises are time-consuming and will challenge a student's best efforts, teachers should be judicious in assigning them. In general, the miscellaneous exercises are not so much a review of the chapter as they are extensions of the ideas of the chapter to new and, we hope, interesting situations.

We have also included at the end of each chapter a set of sample test questions which are in general more typical of the exercises included in the chapter than is true of the miscellaneous exercises.

## Chapter 1

## SETS, RELATIONS, AND FUNCTIONS

This chapter has been written in considerable detail, our assumption being that most of the students using this book during the first year of the SMSG experimental program will not have encountered the ideas and the language of sets before. With a good foundation in algebra, however, it is anticipated that the chapter can be taken rapidly. The ideas are intuitively easy, so that whatever difficulties are encountered will probably arise from the students' unfamiliarity with the symbolism. One of the aims of this chapter (and of the book) is to develop a student's ability to use the language of mathematics and to feel reasonably comfortable when studying ideas represented symbolically. We hope to achieve this in part by including numerous exercises which involve essentially nothing more than algebraic manipulations but which are expressed in the language of sets.

For teachers who wish to study other sources of information dealing with the material of this chapter, we have included on page of this manual a short list of useful references.

Sections 1-6, 1-7, and 1-12 are starred to indicate that they deal with optional topics. They are optional, not because the ideas are difficult, but because they are not essential to the subsequent development of elementary functions. Throughout the book, wherever an exercise depends upon any of this optional material, it is starred.

Although these optional topics are not prerequisite to the rest of the course, we believe that this material would be helpful background information for students planning to continue the study of mathematics in college. Furthermore, the optional material contains interesting mathematics, introducing some of the ideas of an algebra of sets and considerably extending the range of problems available for exercises. It should be observed, also, that students who subsequently take a course in probability and statistical inference, one of the possibilities suggested by the Commission for the second semester of the 12th grade, will need all of the optional material on sets included in this chapter. For these reasons, then, we urge teachers to include this material (particularly Sections 1-6 and 1-7) as part of the course, if time permits, either while studying Chapter 1 or after the regular course has been completed.

The Exercises in this chapter exhibit the wide range of applications of the various concepts discussed in the text and yet do not exhaust them. More than enough problem material is provided to meet the needs of the students. For some of the exercises there will be more than one correct answer or form of the answer and we shall try to indicate this in the solutions for the exercises given in this manual.

In many of the problems, as in the text, we speak of the set of digits, the set of natural numbers, of integers, of rational, and of real numbers. By the set of digits we mean only the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. The set of natural numbers is the infinite set {1, 2, 3, ...}. The set of integers contains the natural numbers, zero, and the negatives of the natural numbers. The rational numbers are an extension of the integers to include numbers of the form p/q, p an integer and q a positive integer. The real numbers include the rational and irrational numbers.

## 1-1. Meaning of Set

The word "set" is used as a universal, collective noun, descriptive of collections of things. We use it because the word is short and easy to say, and because its meaning is independent of the context in which it is used. Thus, in one instance we may be talking about a set of people, in another situation about a set of points, or a set of numbers. Our attention is focussed upon set as an entity in itself rather than upon the particular elements contained in the set. On the other hand, the collective noun "herd" would not serve the purpose of a universal synonym because it is not universal. It implies a collection of cattle, or possibly elephants, but it would hardly be a proper use of the word to talk about a herd of numbers or a herd of points.

We do not pretend to define the term set, but use it in the very natural, commonplace way that one does when talking about the set of books in his library. We are careful, however, to define any particular set in the sense that we specify exactly which elements belong to it.

The notion of a collection of objects is so intuitively obvious that one may well raise the question as to why it has become so important in the study of mathematics today. Oversimplified, the answer is based on the fact that the concept of a set has great unifying power and has become one of the basic concepts in terms of which other mathematical ideas are explained. The modern theory of sets, from which the algebra of sets, point-set topology, and a host of other mathematical subjects have developed, was created by Georg Cantor at the end of the nineteenth century. This theory was needed in order to give a sound foundation for the notion of infinity. "...the concept of the infinite pervades all of mathematics, since mathematical objects are usually studied, not as individuals, but as members of classes or aggregates containing infinitely many objects of the same type, such as the totality of integers, or of real numbers, or of triangles in a plane. For this reason it is necessary to analyze the mathematical infinite in a precise way.... Cantor's theory of sets has met this challenge with striking success. It has penetrated and strongly influenced many fields of mathematics, and has become of basic importance in the study of the logical and philosophical foundations of mathematics. The point of departure is the general concept of a set or aggregate. By this is meant any collection of objects defined by some rule which specifies exactly which objects belong to the given collection."*

Our point of departure in this text is the general concept of a set, but we have no intention of teaching set theory. We study sets for two reasons: (1) because we want our students to become familiar with a language and a mode of thinking which they will encounter repeatedly in the further study of mathematics and (2) because the use of set thinking enables us to deal with conventional school mathematics in ways which we believe are more useful and no more difficult to understand than the traditional treatments. For example, the definition of a function given in this chapter is in current use, is unambiguous, and is not difficult to comprehend if one understands the language of sets.

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*(Courant and Robbins, What is Mathematics?, Oxford University Press, New York, 1941, pages 77-78.)

## 1-2. Describing a Set

Any given set is defined, or described, by specifying exactly which elements belong to it. This is done either explicitly by listing all the elements of the set or implicitly by stating some rule or condition which selects unambiguously the elements of the set. When dealing with small finite sets, the listing or tabulating of all the elements of a set is a convenient and immediately apparent way of describing the set. More commonly, however, we shall be dealing with infinite sets, e.g., the set of all real numbers, and in such cases we must use a rule to describe the set. A rule for defining a set can be almost any statement, however weird, provided it meets the one important condition that it must be possible to decide for any object whether or not it satisfies the rule.

Warning! It is possible to describe sets for which this decision is difficult to make. For example, it is easy to talk about the set of all people in Illinois at this moment, but can you decide whether or not Mr. Thomas James is a member of this set if all you know is that he is driving across the United States via a route through Illinois? A much more subtle and, in fact, contradictory example deals with Harry, the barber in our town, who shaves all those, and only those, in town who do not shave themselves. The question is this: is Harry a member of the set of the people he shaves? (See R. D. Luce, Studies in Mathematics, Volume I, "Some Basic Mathematical Concepts," School Mathematics Study Group, 1959, pages 18-19; also Courant and Robbins, What is Mathematics?, page 87.)

## 1-3. Notation

The braces {} used to enclose the elements of a set call attention to the fact that we are to think of the collection as a single entity. The set-builder notation $\{x : x \ldots\ldots\}$ is a useful way to represent a set which is characterized by some rule or property, nothing more. In some treatments of the subject a vertical bar is used in place of the colon in the set-builder notation. We prefer the colon for typographical reasons.

The listing of the elements, or the rule for selecting elements of a set, may lead to other observations about the set and/or its elements. But these conclusions, even if valid, are not necessarily equivalent to the listing or the rule. For example:

(1) Given the set $S = \{1, 3, 5, 7, 9, 11\}$, one observes that all the elements of $S$ are odd numbers. If this is so, is $13 \in S$? Of course not, since 13 is not displayed as an element of $S$.

(2) The set $T = \{1^4, 2^4, 3^4, 4^4, 5^4, 6^4, 7^4, \ldots\}$ is the same as the set {1, 16, 81, 256, 625, 1296, 2401, ...}, since writing an element in two different ways does not alter the element. Now this infinite set in its second form exhibits a pattern for the unit's digit of the elements -- it is either 1, 5, 6, or 0. So one might conclude that all integers ending in 1, 5, 6, or 0 are elements of the set T. This, of course, is false.

## Exercises 1-3.

In all of the following answers, letters, symbols, and names of people or objects as well as their sequence may be different without making the answers incorrect. It is not necessary always to name a set by means of a capital letter.

## Answers to Exercises 1-3.

1. a) V = {a,e,i,o,u} or V = {= : = is a vowel}

   b) P = {2,3,5,7,11,13,17,19} or

      = {p:p is a prime number less than 20}

   For technical reasons, 1 is not considered as a prime number. For example, its inclusion would raise a difficulty in the unique factorization theorem.

   c) R = {(names of people living in your house)} or

      = {a:a is a person who lives in my house}

   d) T = {3,9,15,21} or {n:n is an odd multiple of 3 and $n < 21$}

e) N = {17,26,35,44,53,62,71,80} or

{x:x is a two-digit integer, the sum of whose digits is 8}

Note: 08 is not considered a two digit number in our system.

2. a) S = {s:s is a student in our school}

b) M = {#:# $>$ 7}

c) P = {p:p is a person in our community who found a ten-dollar bill yesterday}

d) B = {b:b is a book in our school library}

e) F = {f:f is a rational number between 2 and 3}

Here are three types of sets which are not tabulated.

a) and d) represent extensive and lengthy lists which are available somewhere as completely tabulated sets but usually not duplicated.

b) and e) represent examples of sets which contain an endless number of elements and thus defy listing.

c) represents a condition frequently found in mathematics where even though the description is clear and well-defined it still requires a great deal of work or ingenuity to find the elements.

3. a) A = {a:a is a positive even integer less than 12}

or ···· even natural number less than or equal to 10}

b) B = {b:b is an integer whose square is less than 10}

or ················ and $-3 \leq b \leq +3$}

c) C = {c:c is a square of 1,2,3,4 or 5}

or ······ the square of an integer and $0 < c < 26$}

or $\{c^2 : c$ is an integer and $1 \leq c \leq 5\}$

d) $D = \{d : d = 2 + 3n$, n is an integer, and $0 \leq n \leq 5\}$ or .... d is a number of the form $3n - 1$, and $n = 1, 2, 3, 4, 5$ or $6\}$ or ..... d is a term in an arithmetic sequence whose first term is 2, whose common difference is 3, and whose last term is 17}

e) E = {e:e is a permutation of the digits 1,2, and 3} or {abc:abc is a permutation of 123} or {e:e is a three-digit integer formed from the digits 1,2, and 3 without repetition}.

## 1-4. Subsets

It is important not to read into the definition of "inclusion" more than is there. The definition states that the set A is included in (is a subset of) the set B if every element of A is an element of B. In symbols, $A \subset B$. Notice that the definition imposes no restrictions on the elements of B except that they must include all the elements of A. (In this connection it should be mentioned that the symbol $B \supset A$ is sometimes used when we wish to say that the set B *includes* the set A. The set B may then be called a *superset* of A. We have not used this notation in the text.)

From the definition it follows that a set is a subset of itself. If we write $A \subset A$, we mean that every element of set A is an element of set A, obviously a true statement. Likewise, to show that the empty set is a subset of every set, we note that $\emptyset \subset A$ could be false only if the empty set $\emptyset$ contained an element not in A. Since the empty set contains no elements at all, this is impossible no matter what the set A may be.

A few words as to why the empty set is a useful concept may be helpful. We observe that $\emptyset$ is related to the number zero in the same way that the set $\{5, 8\}$ is related to two or $\{7\}$ is related to one. The number of elements in the empty set is zero. But why is such a seemingly vacuous notion useful? It is simply because we do not wish to have to treat as a special case those sets defined in such a way that they contain no elements. For example, when studying quadratic functions we may wish to confine our attention to the set of rational roots of a given equation. Thus, the set of rational roots of the equation $3x^2 + 4x - 4 = 0$ is the set $\{-2, 2/3\}$. But the set of rational roots of the equation $3x^2 + 4x - 3 = 0$ contains no elements, since the roots of the given equation are irrational $(-2 \pm \sqrt{13})$. Hence, the set of rational roots in this case is the empty set $\emptyset$.

By definition the statement $A \subset B$ does not exclude the possibility that $B \subset A$. If both relations hold, then the sets A and B are equal (they both contain exactly the same elements) and we write $A = B$. The equality of two sets may not be obvious at first glance. For instance, the set of digits in the number which is the sum of 8 and 5 is equal to the set of roots of the equation $x^2 - 4x + 3 = 0$. (The set in each case is $\{1, 3\}$.)

The term proper subset is used when we wish to exclude the subset of a given set which is equal to the given set. For example, the proper subsets of the set $\{2, 3\}$ are $\{2\}$, $\{3\}$ and $\emptyset$. Although we have not included the notion of equivalent sets in the text, brief mention of the concept here may be helpful.

Two sets are equivalent if the elements of one set can be put into one-to-one correspondence with the elements of the other set. Another way of saying this is that with each element of one set we can pair an element of the other set in such a way that no element of either set is left over at the end of the process. When dealing with finite sets, this concept leads to the fact that the elements of a given set cannot be put into one-to-one correspondence with the elements of any of its proper subsets. In other words, a finite set is not equivalent to any of its proper subsets.

When dealing with infinite sets, however, it is possible for a set to be equivalent to a proper subset of itself. For example, the infinite set of positive even integers is a proper subset of the infinite set of all positive integers, and the two sets are equivalent. We can show the one-to-one correspondence between the elements of the two sets like this:

$$\begin{array}{ccccccc} 2, & 4, & 6, & 8, & \ldots, & 2n, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1, & 2, & 3, & 4, & \ldots, & n, & \ldots \end{array}$$

(For further discussion of this idea, see Courant and Robbins, What is Mathematics?, pages 78-79.)

Although it might be convenient, we do not have special symbols for disjoint sets nor for sets which "overlap." (Sets A and B overlap if some but not all the elements of A are elements of B and vice versa.)

With regard to the notation $a \in A$ used to indicate that a is an element of the set A, students sometimes tend to confuse $\in$ with the inclusion symbol $\subset$. The latter is used only between sets, whereas $\epsilon$ is used for elements contained in sets. It may be helpful to point out that the Greek letter $\epsilon$ (epsilon) is a symbol for our letter "e," and e is the first letter of the word "element."

Another point of confusion sometimes arises between the symbols $D$ and $\{D\}$. The capital letter by itself is simply a name for a given set, whereas $\{D\}$ stands for a set whose only element is the set D. Also, x and $\{x\}$ may be confused. From page 38 of the 23rd Yearbook* we quote, "Even if A has only a single member, say x, it is not at all true that A is x. ... we can write $A = \{x\}$, but this is quite different from $A = x$. For instance, the Federation of Women's Clubs is a set whose members are clubs, not women. If the Sagebrush Women's Society for Motet Singing has just two members, it can be a member of the Federation. If one of the two women leaves, and only Mrs. Smith is left in the Society, the Society can still belong to the Federation; Mrs. Smith cannot. If Mrs. Smith, discouraged, permits the Society to die, although the Society is the same as the set {Mrs. Smith} it may still be true that Mrs. Smith retains perfect health and vigor."

The ideas discussed in the preceding two paragraphs can be reinforced by consideration of questions like this:

Are the following statements true or false?

a) $\{2\} \subset \{1, 2, 3\}$,  b) $2 \subset \{1, 2, 3\}$,

---

*Twenty-Third Yearbook, "Insights Into Modern Mathematics," National Council of Teachers of Mathematics, Washington, D. C., 1957, page 38.

c) $\{2\} \subset \{2\}$, d) $\{2\} \in \{1, 2, 3\}$,

e) $\{\{2\}\} \in \{\{2\}, 2, 3\}$ f) $\{\{2\}\} \subset \{\{2\}, 2, 3\}$

g) $2 = \{2\}$.

(Answers: (a), (c), (e), and (f) are true;
(b), (d), and (g) are false.)

The concept of a universal set enables us to restrict ourselves in advance to the elements we wish to talk about in any particular discussion. For example, in a first year algebra course we are likely to say that $x^2 - 2$ cannot be written as the product of two binomial factors. This is true only if it is understood that we are restricting the numerical terms in the factors to the universe of rational numbers. In the universe of real numbers $x^2 - 2 = (x + \sqrt{2})\ (x - \sqrt{2})$. As another example, we may say in the same course that the sum of two squares cannot be factored. This is true in the universe of real numbers, but it is false if we agree to admit the complex numbers. Thus, in the universe of complex numbers $a^2 + b^2$ factors into $(a + bi)$ $(a - bi)$.

## Exercises 1-5.

The note under problem 4 is intended for students who have studied combinations. Problem 5 will give the teacher an opportunity to review the various extensions of the number system, if needed. Problem 6 should be assigned with Problems 7 and 9, since the latter are more easily understood in terms of the specific examples in Problem 6.

Answers to Exercises 1-5.

1. a) M = {(Insert 5 names of students in your mathematics class)}

   e.g., M = {Joe, Bill, Henry, Mary, Jean}

   U is the set of all students in your mathematics class.

   or U = {p:p is a member of your mathematics class}

   b) T = {(Insert names of 3 teachers in your school)}

   U is the set of all teachers in your school, district, city, or state.

   c) O = {1,3,5,7,9}. U = {1,2,3,4,5,6,7,8,9} or U is the set of integers (or rational numbers, or real numbers).

   d) F = $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}\}$. U = $\{\frac{a}{b}$:a and b are digits and b $\neq$ 0$\}$ or U is the set of rational numbers.

   e) P = {2,3,5,7,11,13,17,19}.

   U is the set of 1) natural numbers or 2) integers.

2. a) As an example: {QD, KH, JD} or {King of diamonds, Jack of hearts, Queen of hearts} or any of the 18 others. If aces are considered face cards there will be 56 sets.

   b) A = {AS, AH, AD, AC} or {Ace of spades, ace of hearts, ace of diamonds, ace of clubs}

   c) H = {AH, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH}

   d) L = {AH, AD, 2D, 2H, 3D, 3H}

3. a) {2,4} , {4}, {2}. ∅ Four subsets.

   b) {2,3,4}, {2,3}, {2,4}, {3,4}, {2}, {3}, {4}, ∅ Eight subsets.

Answers to Exercises 1-5 (cont'd)

4. a) $\emptyset$

   b) $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$

   c) $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$

   d) $\{1,2,3\}$, $\{2,3,4\}$, $\{1,3,4\}$, $\{1,2,4\}$

   e) $\{1,2,3,4\}$

   f) 16 $(= 2^4)$

   g) If a given set contains $n$ elements, there are $2^n$ possible subsets (including the empty set and the given set).

5. a) $N = \{2, \pi, -3, -\sqrt{3}, \frac{3+\sqrt{2}}{2}, \frac{4}{7}\}$

   b) $Q = \{2, -3, \frac{4}{7}\}$

   c) $P = \{2, \pi, \frac{3+\sqrt{2}}{2}, \frac{4}{7}\}$

   d) $S = \{\pi, -\sqrt{3}, \frac{3+\sqrt{2}}{2}\}$

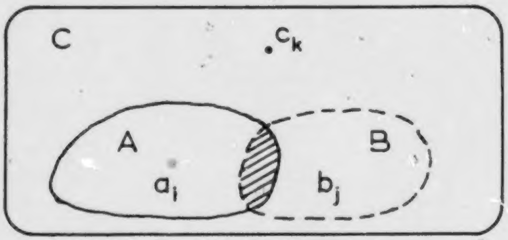
   e) $T = \{\sqrt{-5}, \frac{5-\sqrt{-3}}{3}\}$

6. No: a,e, and f. Yes: b,c,d,g, and h

7. All are yes. See exercise 6 for illustrations. The $c \in C$ and $c \notin A$ and $c \notin B$ is 9. The $c \in C$ and $c \in A$ but $c \notin B$ is either 1 or 7.

8. $A \subset C$.

9. Yes for a,b, and d. No for c. (if shaded area is empty) This Venn diagram may make it clear.



## *1-6 and 1-7. Venn Diagrams and Operations With Sets

As explained in the Introduction, these sections are optional since they contain material not needed in the rest of the course. However, they contain interesting ideas which are mathematically and are not difficult to understand. Even if Section 1-7 is omitted, the inclusion of Section 1-6, Venn Diagrams, would give students a visual way of looking at subsets. For example, to show that {3, 4, 5} is a subset of {1, 2, 3, 4, 5, 6, 7}, you can use a Venn diagram as in Figure TC, 1-6a.

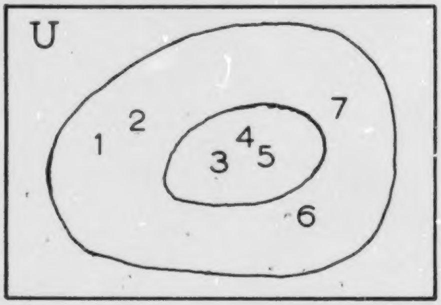


Figure TC,1-6a

The universal set U may be any set you wish, so long as it includes the two given sets. In some cases you can just as well ignore the universal set in the diagram or let the larger of the two given sets be the universe, as in Figure TC, 1-6b.

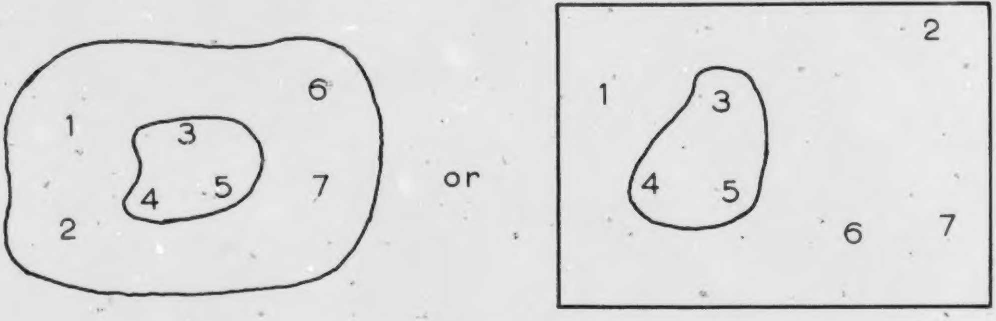


Figure TC,1-6b

In using Venn diagrams it is worth noting that exact circular or rectangular shapes are not necessary, any closed shape being satisfactory.

When operating with sets, Venn diagrams are often useful in pointing the way to solutions, but should not be regarded as proofs of relationships between sets.

The symbol for union is often called cup; for intersection, cap. When thinking about these operations, it helps to keep reminding oneself that the union or the intersection of two or more sets is itself a set. This is especially true when we are dealing with an expression like $A \cup (B \cap C)$, as in Problem 6 of the Exercises 1-7. Here we are looking for the set whose elements are either in A or in both B and C.

## *Exercises 1-7.

These are optional problems covering the material in Sections 1-6 and 1-7. Specific comments are included with the answers to some of the problems. Please note that Problem 6 is intended for those students who are interested in considering some of the ideas in the algebra of sets. It is a fairly difficult problem if formal proofs are required.

### Answers to Exercises *1-7.

1. a) $B' = \{1,2,8,9\}$
   b) $A \cup D = \{1,2,3,6,7,8,9\}$
   c) $B \cap D = \{6,7\}$
   d) $(B \cup D)' = \{1,2\}$
   e) $A \cup \emptyset = \{1,2,3\}$
   f) $C \cap \emptyset = \emptyset$
   g) $(C \cap \emptyset)' = \{1,2,3,4,5,6,7,8,9\}$
   h) $U \cap D = \{6,7,8,9\}$

i) $(U \cap B)' = \{1,2,8,9\}$

j) $(U \cap \emptyset)' = \{1,2,3,4,5,6,7,8,9\}$

k) $A \cap C = \emptyset$

l) $(U \cup A)' = \emptyset$

2. a) $A \cup A = A$

b) $A \cap A = A$

c) $A \cup U = U$

d) $A \cap U = A$

e) $A \cup A' = U$

f) $A \cap A' = \emptyset$

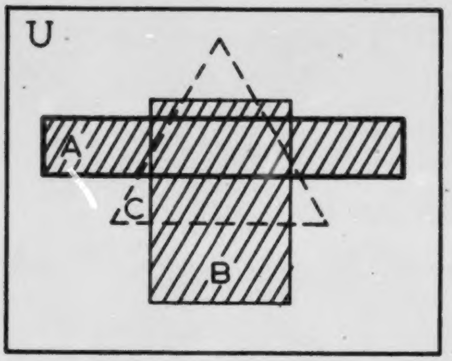
g) $A \cup \emptyset = A$

h) $A \cap \emptyset = \emptyset$

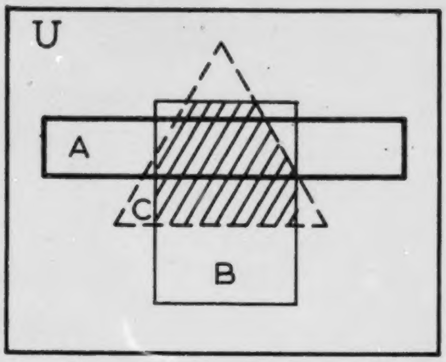
i) $\emptyset' \cap \emptyset = \emptyset$

j) $\emptyset' \cup \emptyset = U$

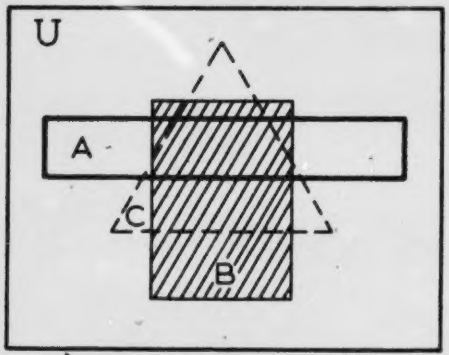
3\.



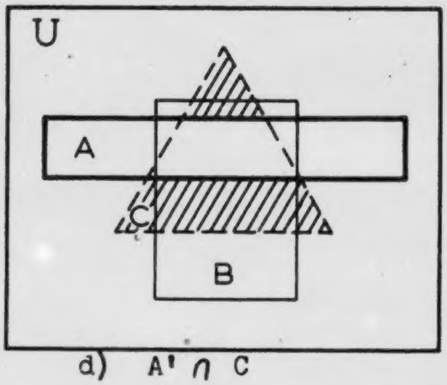
a) $A \cup B$



b) $B \cap C$



c) $B \cap U$



d) $A' \cap C$

## Answers to Exercises *1-7 (cont'd)

4. $S = \{0,1,4,5,6,9\}$

   $S' = \{x : x \text{ is any integer except } 0,1,4,5,6,9\}$

   ($S'$ is an infinite set.)

5. a) If $A' \subset B$ or $B' \subset A$

   b) Only if $A = \emptyset$ and $B = \emptyset$

   c) Only if $B \subset A$ (which holds for $B = \emptyset$)

   d) Only if $A \subset B$

   e) Only if $A = B$

   f) Always

   g) Only if $U = A$

   h) Only if $A = \emptyset$

   i) Always

   j) Only if $U = A$.

   In 5a, if all elements not in one set are in the second set, their union, then, must contain all elements in universe. This is obviously true if either set alone is the universe.

6. Although these theorems may be examined by Venn diagrams, their proof depends on a minute examination of the meanings of each of the symbols $\in$, $'$, $\cup$ and $\cap$. Essentially, all that most students will be able to do is argue the validity of each equation.

## 1-8. Open Sentences

The idea that the familiar equations and not so familiar inequalities of elementary algebra can be regarded as open sentences seems to us to be one of the most helpful concepts to have evolved from recent attempts to improve the secondary school mathematics curriculum. The difference between the statement, "Mr. Jones is my algebra teacher," and the open sentence, "He is my algebra teacher," is readily understood and leads naturally to consideration of the difference between a statement like, "$2 + 5 > 10$," and an open sentence such as, "$X + 5 > 10$." The symbol "x" is called a variable and simply represents any unspecified element of some given set, usually a set of numbers. The given set is the universe, and by substituting one element of the universe at a time for x, students can discover which values of x yield true statements for a given open sentence.

Of course we do not intend to imply that you should confine the problem of solving inequalities to the method of direct substitution, but we believe that this is a useful technique for developing the notion of a variable. Furthermore, it is likely that your students have not previously studied formal methods for solving inequalities, so that as you develop these methods the use of direct substitution as a check will be helpful in building confidence in the techniques.

In the current literature (for example, see the Report of the Commission on Mathematics referred to earlier) the idea that a variable is a "placeholder" is used frequently. For instance, in the open sentence "$X - 4 = 7$," the variable "x" is thought of as holding a place for a numeral which is the name for a number. Although this way of looking at a variable may be helpful at the start, the SMSG courses do not use it. For one thing, a proper development of this point of view requires a rather careful study of the distinction between an object and its name, in particular between a number and its name (called a numeral). Teachers should understand this distinction and may wish to refer to it briefly in class discussion, but it is our feeling that undue stress on this point is not needed for understanding the role of a variable. If the occasional discussion arises in which the difference between a number and its name is relevant, the matter can always be settled on the spot.

In summary, we believe that the ideas to stress are the following: (1) equations and inequalities are open sentences containing one or more variables (open sentences in two variables are studied later in this chapter), neither true nor false as they stand; (2) a variable is a symbol used to represent any unspecified element of a given set; and (3) the given set is the universe. The fourth important idea is discussed in the following section.

## 1-9. Solution Sets of Open Sentences

The solution set of an open sentence is that subset of the universal set which contains all elements for which the open sentence becomes a true statement. In other words, an open sentence selects a subset of the universe, namely the solution set, and for this reason open sentences are sometimes called "set selectors."

A large part of this section is devoted to illustrative examples, since this is our first opportunity to show how the language of sets can be adapted to ideas of elementary algebra. Free use is made of the number line for graphing the solution sets of open sentences in one variable, and we have adopted certain symbols, explained in the text, for indicating when a particular point representing a number is or is not an element of the solution set.

Concerning the graph in Figure 1-9a of the text, it would probably have been better to use a dashed line rather than the solid line, since in this example the universe is a small finite set of integers. A solid line tends to imply that any point on it represents a number in the universal set; in other words, that the universe is the set of real numbers. This is not the case in this example, and we would favor modifying the graph as shown in the following Figure TC, 1-9a:

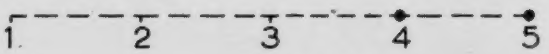


Figure TC, 1-9a. Graph of $x > 3$ when $U = \{1,2,3,4,5\}$.

Although the primary aim of this section is to illustrate how the language of sets may be used with algebraic material, a secondary purpose is to provide a review of certain algebraic techniques and an extension of the ideas to include the solution of inequalities, the notion of absolute value, and so forth. If in this course the solution of inequalities is being studied for the first time, it will be necessary to extend the set of axioms used in solving equations to include operations on inequalities. This was probably done to some extent when your students studied geometry, but it is unlikely that negative numbers were included. For example, what conclusion is implied by the operation of multiplying each member of $x < 3$ by $-2$? The answer is that $-2x > -6$.

For your convenience we state here the general rules used in solving inequalities: (1) The solution set of an algebraic inequality is not altered by adding the same term to or subtracting the same term from both members, or by multiplying or dividing both members by the same positive term; (2) If both members of an inequality are multiplied or divided by the same negative term, the sense (direction) of the inequality is reversed. For example, $-2 < 3$, but if we multiply both members of the inequality by $-4$, we obtain $8 > -12$. A much more subtle example, harder than any that appear in the text, is the following: In the universe of real numbers find

$$\{x : \frac{3}{x - 2} < -4\}.$$

We wish to clear the inequality of fractions so we multiply both members by $x - 2$. But is the term $x - 2$ positive or negative? By inspection of the given inequality we see that the fraction $3/(x - 2)$ must be negative and hence $x - 2$ must be negative. Therefore, multiplying both members by this term, we obtain

$3 > -4x + 8$. (Note the reversal of the inequality sign.)

Subtracting 8 (or adding $-8$) gives

$$-5 > -4x,$$

and dividing by $-4$, we have

$$\frac{5}{4} < x.$$

(Why was the inequality sign reversed again?)

Making use of the fact that if $a < b$, then $b > a$, we can write the last result as

$$x > \frac{5}{4} .$$

But there is a second condition that must be satisfied. We stated above that the term $x - 2$ must be negative. This is the same as writing

$$x - 2 < 0,$$

and from this it is clear that $x < 2$. Hence, the solution set for this example is defined by an open sentence which imposes both a lower and an upper limit on the values of x. We write as the final answer

$$\{x : x \in R \text{ and } 5/4 < x < 2\}.$$

(Note that R is a symbol for the set of real numbers.)

As an exercise for teachers we suggest the following problem: what is the solution set defined by the inequality $3/(x - 2) > - 4$ in the universe of real numbers?

(Answer: $\{x : x \in R$ and $x < 5/4$ or $x > 2\}$.)

The concept of absolute value is introduced in this section (see Example 6) and it is used in succeeding chapters of the book. In addition to the immediate comments, further discussion of absolute value will be found in this manual under Sections 2-2 and 3 of Chapter 2.

If x is a real number, the absolute value of x, symbolized by $| x |$ is defined by

$$| x | = \begin{cases} x, & \text{if } x \geq 0, \\ - x & \text{if } x < 0. \end{cases}$$

Thus, if $x = 4$, $| x | = 4$; if $x = 0$, $| x | = 0$; if $x = - 4$, $| x | = - (- 4) = 4$. In other words the absolute value of any number or algebraic expression is always either zero or a positive quantity.

The ideas involved in the solution of open sentences containing absolute values may not come easily, and it would be advisable to begin by discussing the simplest examples, such as $| x | = 5$, $| x | < 5$, $| x | > 5$, $| x | < - 5$, and so forth. The solution set in the last example is the empty set, since $| x |$ is never less than zero.

We shall consider the example $| x | < 5$ in detail. By definition, this means that if x is zero or positive, then $x < 5$, and if x is negative then $- x < 5$, from which it follows that $x > - 5$. Hence, the solution set of the open sentence $| x | < 5$ is

$$\{x : x \in R \text{ and } - 5 < x < 5\},$$

which is the concise way of saying that x must be greater than $- 5$ and less than 5, or that x may be any real number between, but not including $- 5$ or 5.

The outcome of a discussion with examples like the above might well be a set of generalizations such as:

if $| x | = a$, then $x = a$ or $x = - a$;

if $| x | < a$, then $- a < x < a$;

if $| x | > a$, then $x > a$ or $x < - a$;

if $| x + b | \leq a$, then $- a \leq x + b \leq a$, from which we obtain $- a - b \leq x \leq a - b$ by adding $- b$ to each member of the inequality.

Two further examples of open sentences containing absolute values follow. Considering first an equation, if we are given that $| x - 2 | = 3$, then either $x - 2 = 3$ or $-(x - 2) = 3$, and hence $x = 5$ or $-1$. To verify that $\{-1, 5\}$ is the solution set for the given equation we can use direct substitution. If $x = 5$, then $| 5 - 2 | = | 3| = 3$, as required; if $x = -1$, then $| -1 - 2 | = | -3 | = 3$, and our checking is finished.

The use of the number line is helpful when discussing open sentences containing absolute values. Referring to the equation $| x - 2 | = 3$ of the preceding paragraph, we may think of the quantity $| x - 2 |$ as the distance on the number line between the points whose values are x and 2. Since $| x - 2 |$ equals $| 2 - x |$, x may be either to the right or the left of 2 at a distance of 3 units from 2. This leads immediately to the conclusion that x must be 5 or - 1. Figure TC, 1-9b illustrates the idea.

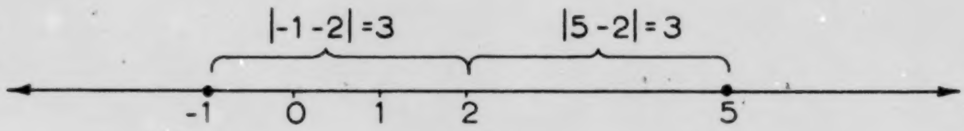


Figure TC, 1-9b. Graph of the solution set for $x \in R$ and $| x - 2 | = 3$.

Inequalities containing absolute values are a little more difficult at first, but here again the use of the number line is instructive. For example, what is the solution set of $\{x : x \in R \text{ and } | x - 2 | < 3\}$? Interpreted graphically, the the inequality states that the distance from x to 2, or from 2 to x, must be less than 3. Since we have already found that when the distance equals 3, x is either - 1 or 5, it follows that for the distance $| x - 2 |$ to be less than 3, x must be to the right of - 1 and to the left of 5. In other words, the solution set is

$$\{x : x \in R \quad \text{and } - 1 < x < 5\}.$$

The algebraic manipulations needed to solve this inequality also depend upon the two alternative conditions implied by absolute value. If $| x - 2 | < 3$, then $x - 2$ may be either positive or negative (or zero). If it is positive, then $x - 2 < 3$, whereas if it is negative, then $- (x - 2) < 3$ or $x - 2 > - 3$. Combining these two results we have

$$- 3 < x - 2 < 3,$$

and by adding 2 to each member of this inequality we obtain

$$- 1 < x < 5.$$

This result agrees with that of the preceding paragraph.

It may be helpful to observe that occasionally students obtain erroneous results because they confuse $- (x - 2) < 3$ with $x - 2 < - 3$. If you are careful, however, to develop the idea that $| x - a | < b$ means that $x - a$ may be positive or negative, but that to write $x - a < \pm b$ is incorrect, then your students ought not to be confused by this distinction.

Closely related to the notion of absolute value is the frequently misunderstood convention regarding the square root symbol $\sqrt{\phantom{x}}$. In fact, either one could be used to define the other since the two definitions are exactly the same. Thus, if x is a real number, then $\sqrt{x^2}$ is x when x is zero or positive and - x when x is negative. In other words, $\sqrt{x^2} = | x |$. For example, $\sqrt{3^2} = 3$ and $\sqrt{(-3)^2} = 3$. Also, $-\sqrt{3^2} = -3$ and $-\sqrt{(-3)^2} = -3$. But note carefully the difference between $\sqrt{(-3)^2} = 3$ and $\sqrt{-3^2}$. The latter is the same as $\sqrt{-9}$ and this is the imaginary number $3i$.

When solving inequalities such as $x^2 > 9$ it is advantageous to use the fact that $\sqrt{x^2} = |x|$. We may then write $|x| > 3$, from which it follows that $\pm x > 3$, and this in turn gives $x > 3$ or $x < -3$. This practice will help students to avoid the rather natural error of assuming that because $x^2 = 9$ means that $x = \pm 3$, $x^2 > 9$ must mean that $x > \pm 3$. It might be helpful to emphasize that $x^2 = 9$ implies that $|x| = 3$ and from this we obtain $\pm x = 3$ and finally $x = \pm 3$. The solution of the inequality $x^2 < 9$ follows the same pattern.

## Exercises 1-9.

Problem 1 is designed to give students practice in writing in set-builder notation open sentences which are expressed verbally. Problem 2 contains the real meat of this section and is designed in such a way that each part contributes to understanding of the parts which follow. These problems take time, and it would be a mistake to expect that your students can do all of Problem 2 in one assignment. The drawing of the graphs of the solution sets should emphasize the mathematics involved rather than the art work. Problems 3-6 are optional since they depend upon Section 1-7 as well as the present section.

Answers to Exercises 1-9.

1. a) $\{x:x > 7\}$. U may be the set of 1) natural numbers, or 2) digits, or 3) integers, etc. It is also possible to include the universe in the set-builder, e.g., $\{x:x \text{ is an integer and } x > 7\}$

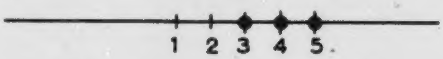
   b) $\{x:x \in R \text{ and } x^2 = 3\}$

   c) $\{x:x \in R \text{ and } x \leq 0\}$

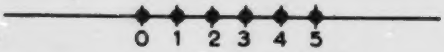
   d) $\{x:x \in R \text{ and } \sqrt{x} > 5\}$ or $\{x:x \text{ is a natural number and } x^2 > 25\}$

   e) $\{x:3\ |x| - 5 = -\ 3\}$, U is the set of real numbers.

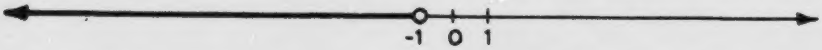
2. a)



b)



c)



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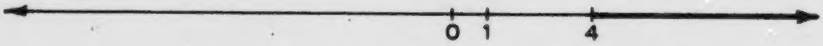

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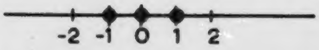

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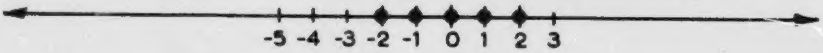

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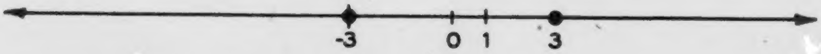

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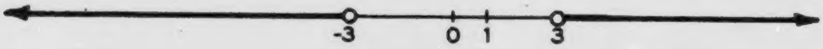
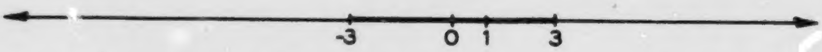
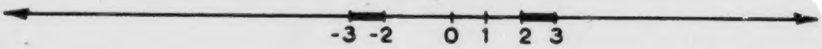


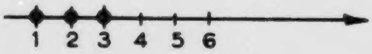
i)



j)



k)



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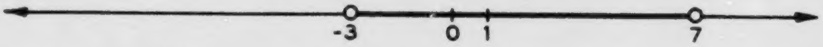

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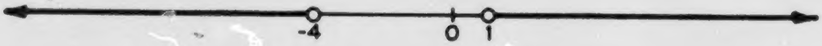

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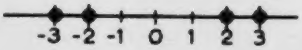

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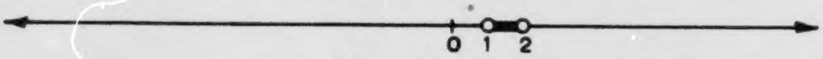


3.* a) $S_1 = \{1,2,3\}$

b) $S_2 = \{3,4,5\}$

c) $S_2' = \{4,5\}$

d) $S_2' = \{1,2\}$

e) $S_1 \cup S_2 = \{1,2,3,4,5\}$

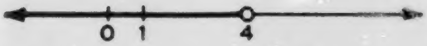
f) $S_1 \cap S_2 = \{3\}$

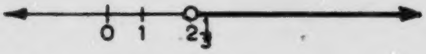
g) $S_1' \cup S_2' = \{1,2,4,5\}$

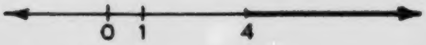
h) $(S_1 \cap S_2)' = \{1,2,4,5\}$

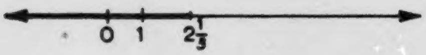
i) $(S_1 \cup S_2)' = \emptyset$

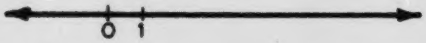
4.*

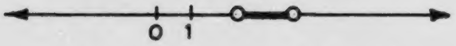
a) 

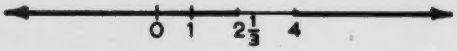
b) 

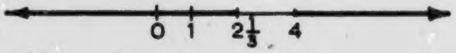
c) 

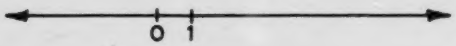
d) 

e) 

f) 

g) 

h) 

i) 

Note: g and h are the same graph.

5.* a) $S_3 = \{-5, -4, -3, -2, 4, 5\}$

b) $S_4 = \{-3, -2, -1, 0\}$

c) $S_3 \cup S_4 = \{-5, -4, -3, -2, -1, 0, 4, 5\}$

d) $S_3 \cap S_4 = \{-3, -2\}$

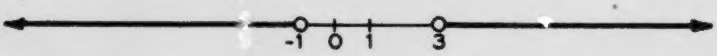
e) $S_3' = \{-1, 0, 1, 2, 3\}$

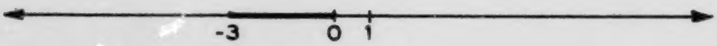
f) $S_4' = \{-5, -4, 1, 2, 3, 4, 5\}$

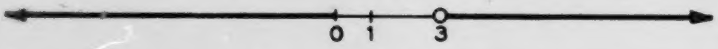
g) $S_3' \cap S_4' = \{1,2,3\}$

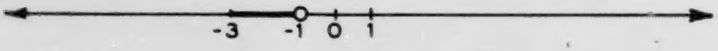
h) $(S_3 \cup S_4) = \{1,2,3\}$

6.*

a) $S_3$ 

b) $S_4$ 

c) $S_3 \cup S_4$ 

d) $S_3 \cap S_4$ 

## 1-10. Ordered Pairs and Graphs

The concept of an ordered pair is an extremely useful one in many branches of mathematics. In addition to the uses in the text (Cartesian products, relations and functions), ordered pairs are used to represent negative numbers, rational numbers, complex numbers, vectors and numerous other mathematical concepts. The student should be quite familiar with the ordered pair idea, though perhaps on an intuitive level only. You might ask a student for the name of the first president of the United States, and point out that this is an ordered pair, with George as the first component and Washington as the second. Or, again, you might ask what an umpire means when during a baseball game he says that the "count" is 3 and 2. Since balls are announced before strikes, this constitutes an ordered pair. You can probably think of many more such examples and should use as many as seem necessary to convince the students that this is a commonplace and extremely simple idea.

We use the word "member" on page 32 to define the components of an ordered pair. Technically this is not good usage since an ordered pair is not a set consisting of its components, but we feel that our meaning is clear, and from this point on, $x$ and $y$ are always referred to as components of ordered pairs.

We have avoided the use of the idea of an independent and dependent variable in our work since the use of sets in discussing relations and functions renders them superfluous.

## Exercises 1-10.

These exercises vary in purpose. Exercises 1, 2, and 3 deal with some simple properties of ordered pairs, while exercises 4 and 5 are largely intended as problems in reading and understanding the words we use to discuss sets of ordered pairs. Exercises 6 to 9 give practice in finding elements of the solution sets for open sentences in two variables and in graphing the sets.

### Answers to Exercises 1-10.

1. $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

2. $(x,y) \neq (u,v) \longrightarrow x \neq u$ or $y \neq v$ or both.

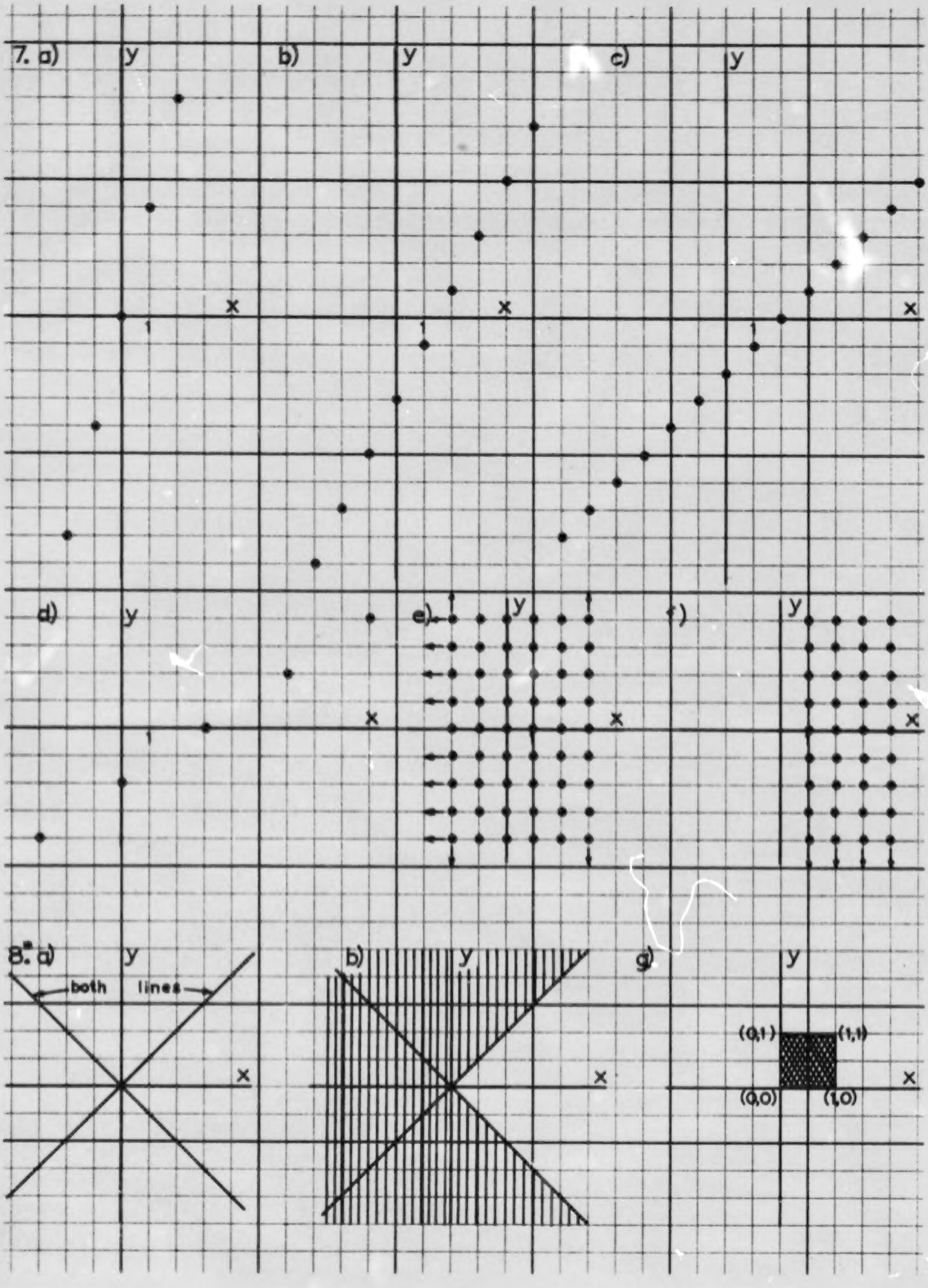
3. a) $x = 3$ and $y = 2$
   b) $x = y$
   c) $y \neq 2$
   d) $y \neq 4$

4. $\{(1,1), (2,7), (3,5), (4,8), (5,3), (6,2), (7,4), (8,6)\}$

5. a) $\{(2,7), (4,8), (5,3), (7,4)\}$ (Braces not essential)
   b) Three: $(2,7),(4,8),(5,3)$
   c) Five: $(1,1),(5,3),(6,2),(7,4),(8,6)$
   d) Yes $(5,3)$
   e) Yes

6. a) $\{(0,0),(7,7)\}$
   b) $\{(3,2),(4,1),(2,1)\}$
   c) $\emptyset$
   d) $\{(4,1),(0,0)\}$
   e) $\{(3,2),(4,1)\}$
   f) $\{(0,0)\}$
   g) $\{(0,0),(2,1),(4,1)\}$
   h) $\{(2,1),(3,2)\}$

7-8. See next sheet for the answers.


7. a)
y
x
1
b)
y
x
1
c)
y
x
1
d)
y
x
1
e)
y
x
f)
y
x
8. a)
y
both
lines
x
b)
y
x
g)
y
(0,1)
(1,1)
(0,0)
(1,0)
x

## 1-11. Cartesian Products

The Cartesian product is used in this text to give a well defined universe for sets of ordered pairs. In particular, R x R is in one-to-one correspondence with the points of the geometric plane and thus serves well for the representation of solution sets of open sentences in two real variables. As mentioned in Exercise 10, the concept can be extended to ordered triples, or indeed, to ordered n-tuples, and appropriate use can be made of the corresponding Cartesian products. For instance, with ordered triples (x,y,z), we can associate the Cartesian product R x R x R with the geometric points in Euclidean 3-space.

In some textbooks, the words "Cartesian set" are used to represent the Cartesian product of a set with itself; R x R is thus said to be the Cartesian set of R . We feel that the notion is not important enough to bring to the student's attention now. Another way to picture the formation of a Cartesian product is by means of the "tree", as in Figure TC, 1-11 where we form the Cartesian product of {1,2,3} with itself.

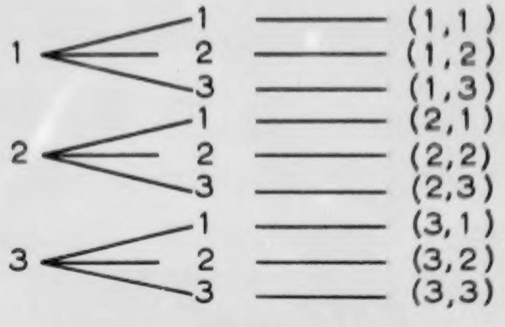


Fig. TC, 1-11

In a sense, the Cartesian product is a mathematician's canvas upon which he is able to picture relationships by selecting various subsets upon which to focus his attention.

## Exercises 1-11.

The exercises are straightforward and designed to familiarize the student with the simpler aspects of Cartesian products, and again, to provide practice in the reading and writing of set symbols.

## Answers to Exercises 1-11.

a) One. $U \times U = \{(1,1)\}$

b) Four. $U \times U = \{(1,1),(1,2),(2,1),(2,2)\}$

c) Nine.

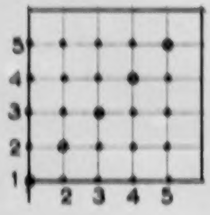
d) $n^2$.

2. $A = \{7,8,9\}$

$$A \times B = \begin{Bmatrix} (7,1),(7,2),(7,3),(7,4) \\ (8,1),(8,2),(8,3),(8,4) \\ (9,1),(9,2),(9,3),(9,4) \end{Bmatrix}$$

$B = \{1, 2, 3, 4\}$

3. a)

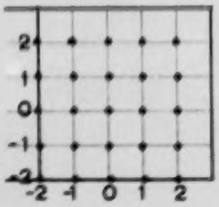


b) $\{(1,2),(2,2),(3,2),(4,2),(5,2)\}$

c) $\{(3,1),(3,2),(3,3),(3,4),(3,5)\}$

d) 25

4.



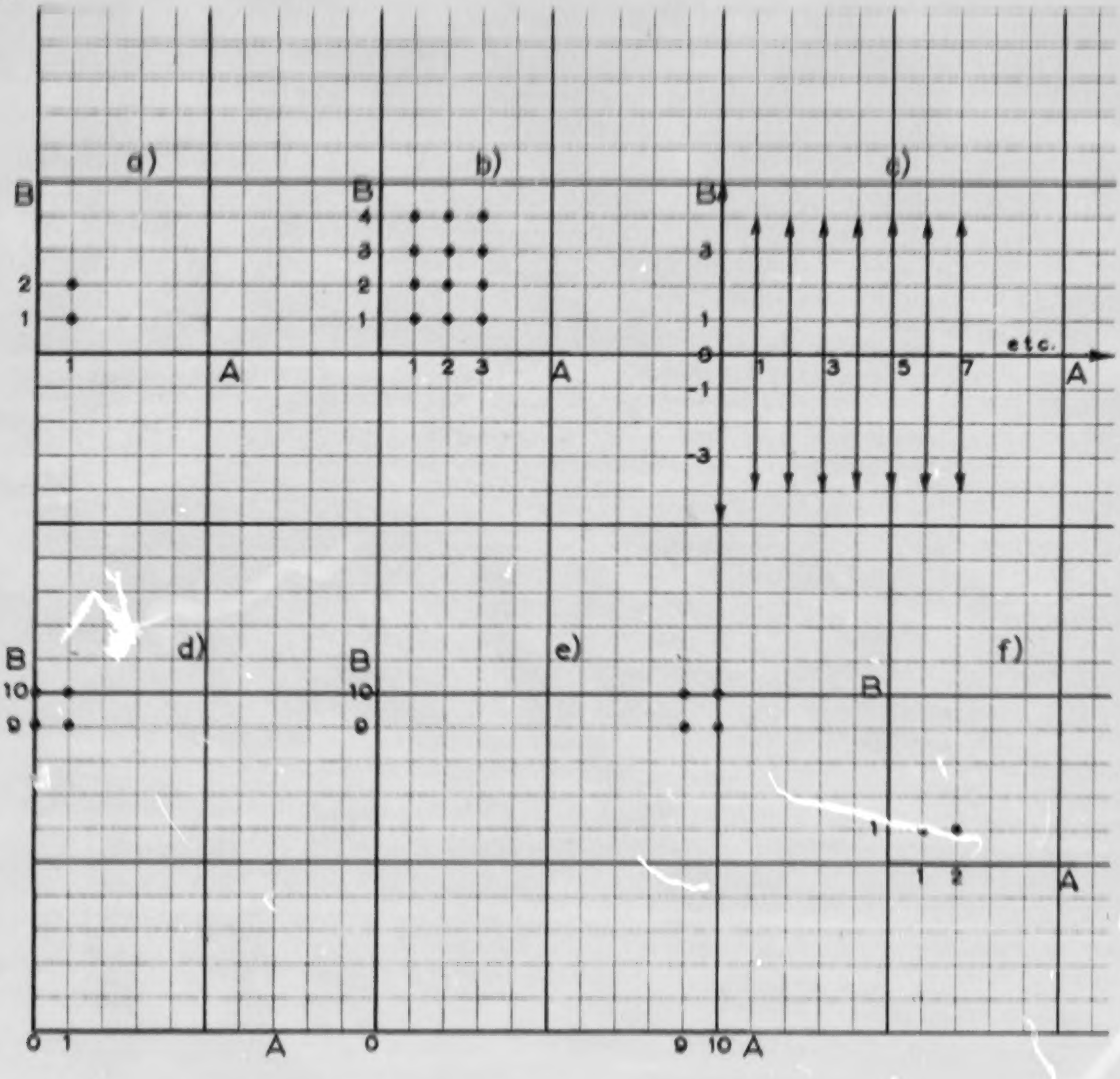
5. a) $A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

   b) $B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$

   c) Six

   *d) Zero

6.



7. a) $mn$

b) $m$ or $n$, whichever is less

c) Yes. Either $m = 1$ or $n = 1$, the other equals 17.

d) Eight. $n = 1,2,3,4,6,8,12,$ or $24$ and $m = \frac{24}{n}$.

8. a) $n$

b) $\frac{n^2 - n}{2}$

c) $\frac{n^2 - n}{2} + n = \frac{n^2 + n}{2}$

9. If $A \subset U$ and $B \subset U$, then $A \times B \subset U \times U$

10. a) $\{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)\}$

b) The vertices of a cube. A lattice is the intersection of perpendicular sets of equally spaced parallel lines. It is a set of points.

c) None for $U = \{1,2\}$

d) $2^4 = 16$

11. a) $\{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$

b) Six

12. $n^2 - n$

13. a) $B = \{6n : n \text{ is a digit and } 6n < 25\}$

$= \{6,12,18,24\}$

$A = \{4n : n \text{ is a digit and } 4n < 25\}$

$= \{4,8,12,16,20,24\}$

$$A \times B = \left\{ \begin{array}{l} (4,6),(4,12),(4,18),(4,24), \\ (8,6),(8,12),(8,18),(8,24), \\ (12,6),(12,12),(12,18),(12,24), \\ (16,6),(16,12),(16,18),(16,24), \\ (20,6),(20,12),(20,18),(20,24), \\ (24,6),(24,12),(24,18),(24,24) \end{array} \right\}$$

b) Deleted $A \times B$ = same as answer to a) but without (12,12) and (24,24).

*c)

$$A \times A \cap A \times B \left\{\begin{matrix}(4,12),(4,24),\\ (8,12),(8,24),\\ (12,12),(12,24),\\ (16,12),(16,24),\\ (20,12),(20,24),\\ (24,12),(24,24)\end{matrix}\right\}$$

## *1-12. Locus

We have starred this section because we make heavy use of the union and intersection of solution sets, and union and intersection are themselves optional topics. We feel, however, that it will prove very popular with those students who cover it since most of the graphs of inequalities and absolute value relationships will be different from those to which they are presently accustomed.

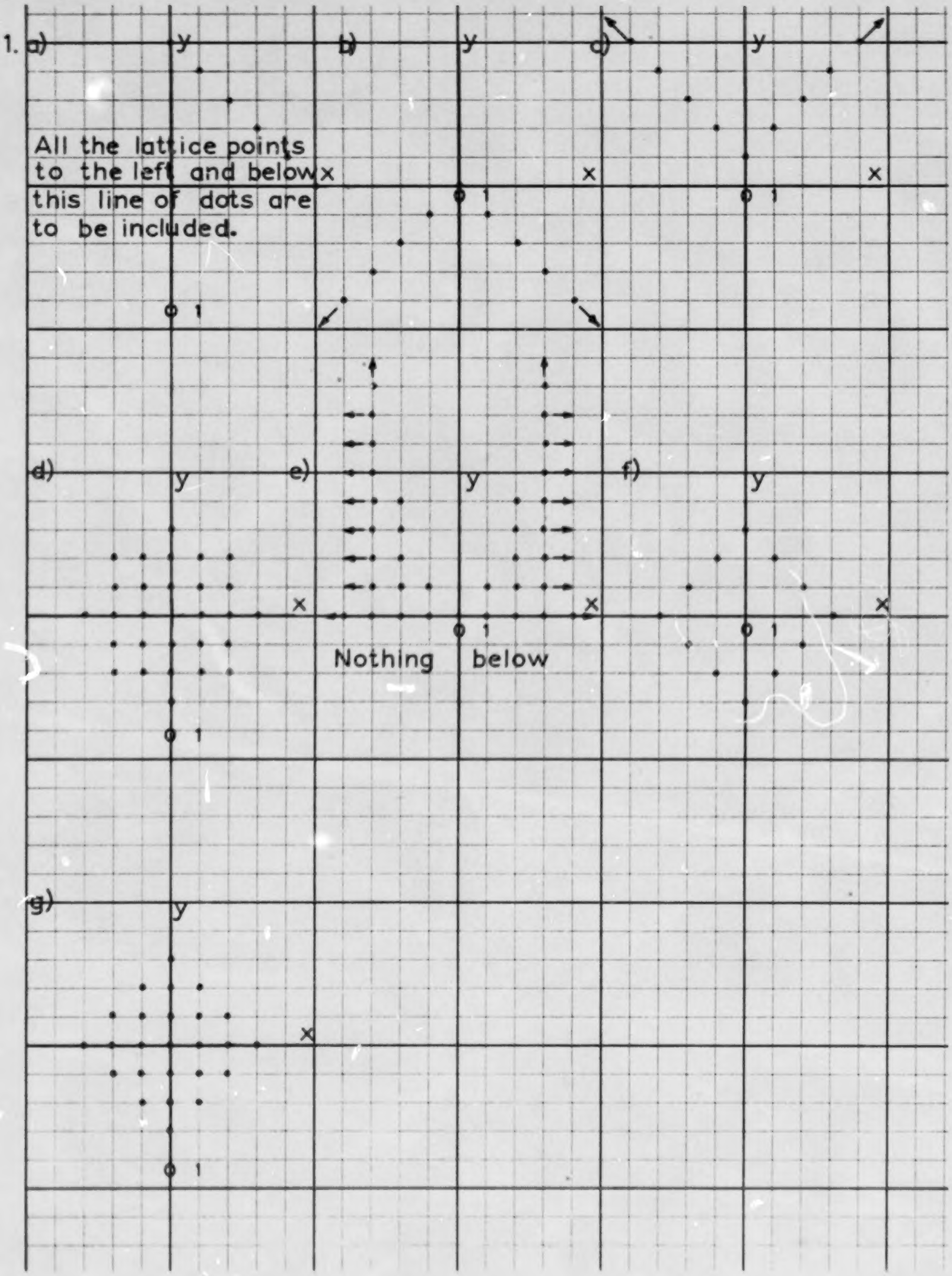
Some attention must be given to the difference between $<$ and $\leq$, and some convention adopted with respect to how to graph the former. The use of the dotted line to indicate that the line is excluded is useful here, but you should feel free to establish another convention if you find it more convenient. The important thing is that the line be included or excluded as the situation requires.

Short cuts to graphing should be encouraged and the student should be urged to look for them. For instance if he has graphed $y = |x|$ and is asked to graph $y = |x| + 1$, the process of plotting a large number of points, or worse, of making up a new table from which to plot them, is a waste of time. Instead, he should be required to ask himself just what effect the addition of the constant 1 will have on the original graph and proceed from there. Furthermore, you should explain how, having graphed $y = |x|$, he can reduce the problem of graphing $y < |x|$, $y \geq |x|$, $y = 3\,|x|$, etc., to a simple matter of sketching, and that he can obtain such graphs almost at once. Every reasonable opportunity to obtain a graph without plotting points should be exploited fully. We have kept the types of graph to a very small number, so that the student can learn to recognize familiar forms and can save himself many hours of laborious computation.

### Exercises 1-12.

The exercises are simply to provide practice in graphing. They may however be extended. For instance, Exercises 5, 6, and 7 suggest the idea that there are set-builders which will provide a graphical pattern similar to that formed by the black squares on a checkerboard. If you wish you may ask the student to find them. In general, the student should be encouraged to experiment with various set-builders to find some which will give interesting patterns.

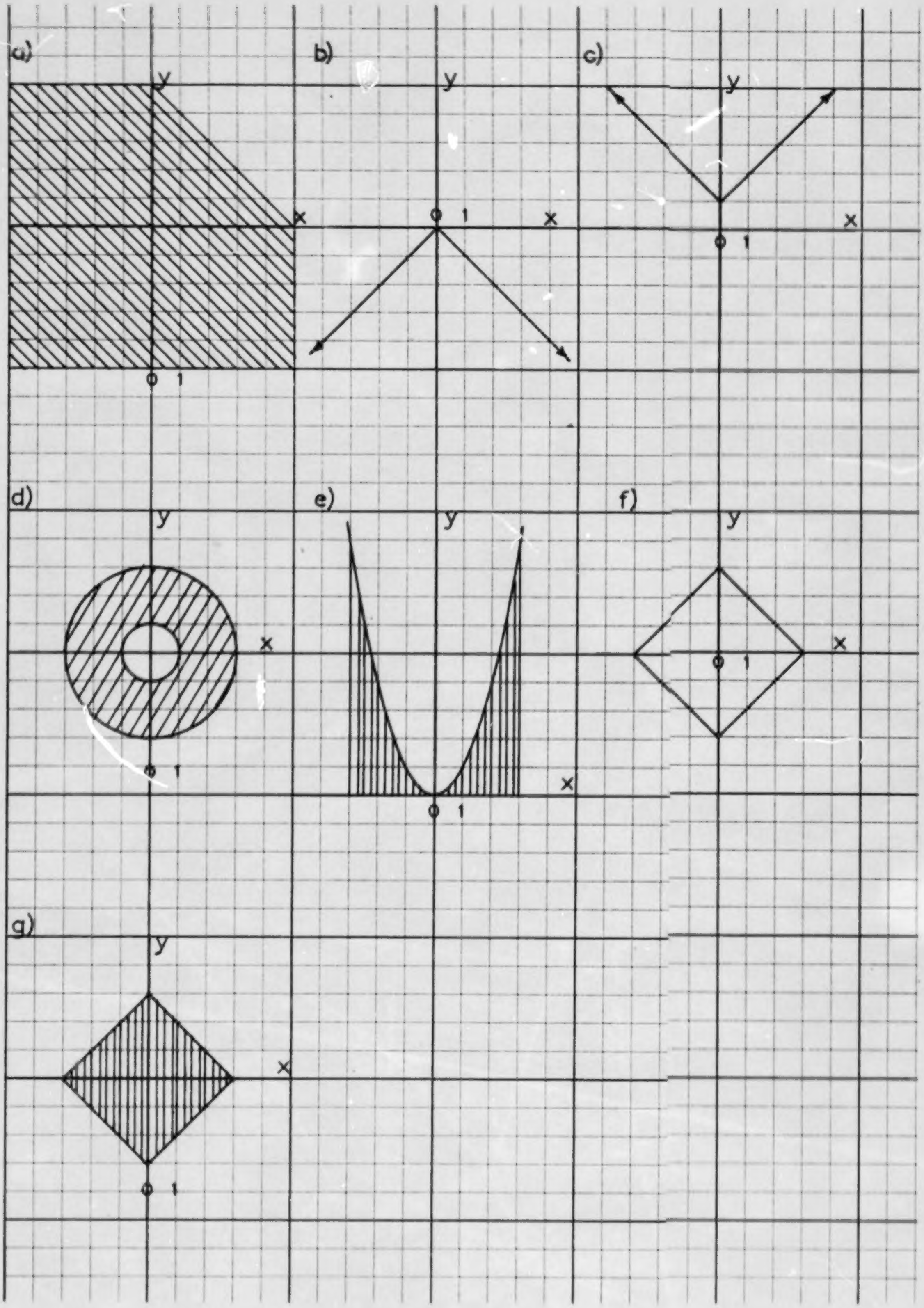
## Answers to Exercises 1-12



All the lattice points to the left and below this line of dots are to be included.

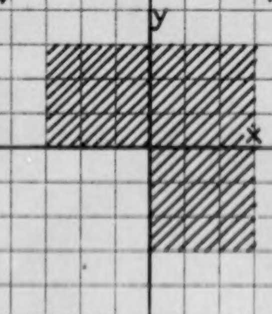
Nothing below

2.

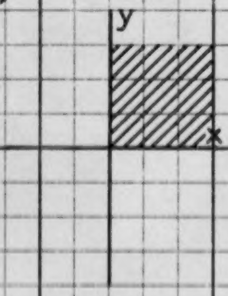


Where continuous lines form boundaries, they are included in the set; dotted boundaries are not included in the set.

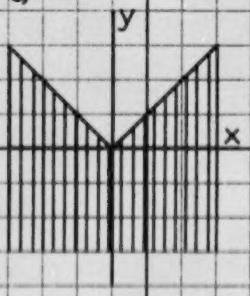
3. a)



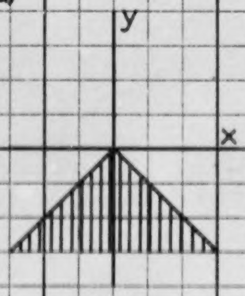
b)



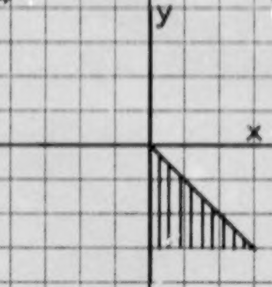
c)



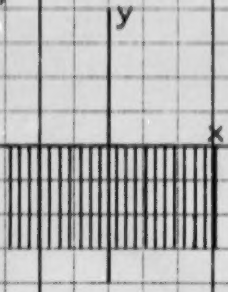
d)



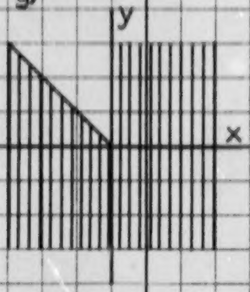
e)



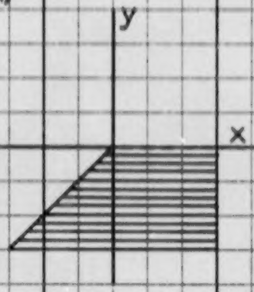
f)



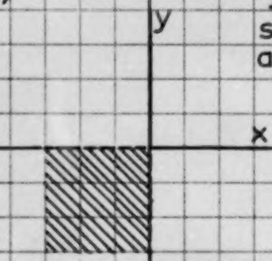
g)



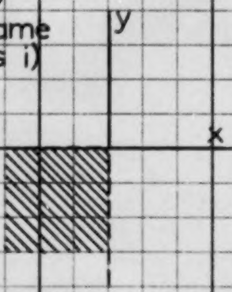
h)



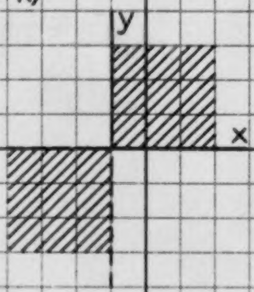
i)



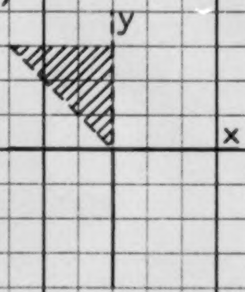
j) same as i)



k)

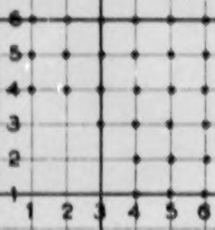


l)

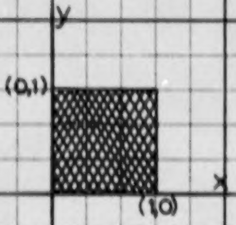


6. $S = \{(x, y) : 1 \leqslant x \leqslant 2 \text{ and } 1 \leqslant y \leqslant 2\}$

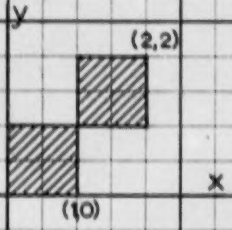
4.



5.



7.



## 1-13. Relations

In the text, we confine our attention to numerical relations, although you may wish to introduce the relation concept with some every day non-numerical examples. The husband-wife, father-son, employer-employee, or doctor-patient relationships (to name a few) are suitable for the purpose. You may go on to show that these have something in common with numerical relations, namely, the association of pairing of the elements of one set with those of another. We may look to geometry for other examples of relations. The parallelism of lines, for instance, is a relation, with any two lines in the plane being either in or out of the relation. The perpendicularity of lines, the incidence of points on lines, congruence properties, etc., are further examples of geometric relations.

Any equation in two variables defines a relation, i.e., its solution set in $U \times U$. If $U$ is finite and contains n elements, then there are exactly $2^{n^2}$ possible sub-sets of $U \times U$ (counting $\emptyset$ and $U \times U$) and hence $2^{n^2}$ possible relations in $U \times U$. In the event that $U$ is an infinite set, there are an infinite number of possible relations in $U \times U$.

In general, any set of ordered pairs is a relation, because we can always obtain a Cartesian product $A \times B$ where $A$ is the set of first and $B$ of second components of the ordered pairs. The given set of ordered pairs is clearly a sub-set of $A \times B$. The reason we define a relation as a sub-set of $A X B$ is to give some specific frame of reference upon which the student may focus his attention. The sets $A$ and $B$ are not referred to as the domain and range of the relation in this text because we wish to emphasize these terms in connection with functions where they are of greater importance and where the distinction is more useful.

One particularly important aspect of this section is the fact that a relation may be represented in various ways. Whether it be specified by graph, set-builder, table, or listing, we wish it understood that we are talking about only one thing, the set of ordered pairs which is the relation. The graph, of course, is usually the easiest way for the student to picture a relation, and we have tried to stress this idea in the exercises. There are relations for which the open sentence is the easiest description: for example, "$x$ and $y$ are incommensurable" (i.e., the ratio of $x$ to $y$ is irrational). Such relations are probably too difficult to treat at the high school level, however, since the student cannot picture them.

There are two other items which might prove of interest. First, the inverse of a relation is obtained by interchanging the $x$'s and $y$'s in each pair in the relation. We have not treated the inverse of a relation in the text because, once again, the idea is more important when applied to functions and we do not wish to have to distinguish between the ideas. Second, if the students have covered Section 1-7, it might be pointed out that since the complement of a subset of $U \times U$ is also a subset of $U \times U$, the complement of any relation in $U \times U$ is a relation in $U \times U$.

## Exercises 1-13.

The exercises are designed as much for practice in graphing as for anything else. In Exercise 1, the idea is to see that a set containing an element which is not in $U \times U$ is not a relation in $U \times U$.

Exercise 2 will require some explanation. We have tried to pick relations which are fairly obvious, since it is certainly outside the province of this text to develop specific machinery for arriving at defining equations for these relations. If you find the exercise too hard for most students, simply omit it.

Exercise 3 may help the student to guess certain parts of 2 if he has not previously found the requisite equation, although in this small finite universe, the picture is not always too suggestive.

Exercise 4 emphasizes graphing again. 4(e) and 4(g) may lead to a discussion of the graphs of factorable equations if you wish to devote any time to it.

Exercises 5 and 6 ask the student to test his understanding of the definitions of a relation and a Cartesian product, while 7 and 8 are designed to show that the universe is very important in the relation concept.

## Answers to Exercises 1-13.

1. The following do not: b - because of zero, c - because of negative integer, e - 12 is not in set and neither are the fractions, f - 10 is not in set, i - fractions, k - negative integers, and n - because if $x$ is any digit, $10x$ is not.

2. a) $\{(x,y): y = 3x\}$ h) $\{(x,y: y = 7x - 6$

b) $\{(x,y): x = 3y\}$ or $y = x^2 + 4x - 4$

c) $\{(x,y): x + y = 10\}$ or $y = x^3)\}$

Note: In set builder notation we may use elements not in our universe as part of our open sentence.

d) $\{(x,y): x = y\}$ i) $\{(x,y): x - 3 = y\}$

e) $\{(x,y): x + 3 = y\}$ j) $\{(x,y): x + 7 = y\}$

f) $\{(x,y): y = 2x\}$ k) $\{(x,y): x = y^2$ or $(y = \sqrt{x})\}$

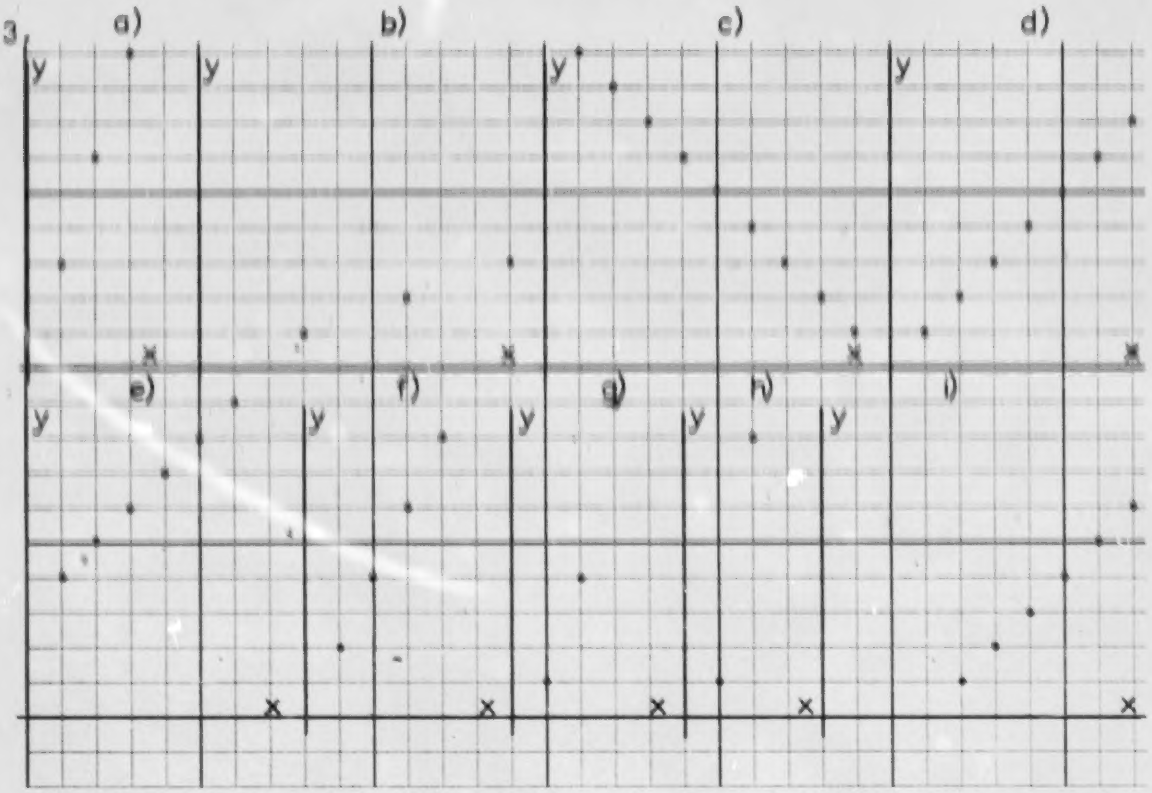
g) $\{(x,y): y = x^2\}$ l) $\{(x,y): y = 2x + 1\}$

m) $\{(x,y): y = 5x - 7$

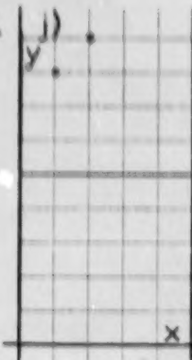
or $(y = x^2 - 1)\}$

Note: This is not intended as material to which curves are to be fitted. If any answer is given which is correct, accept it and continue without discussion.

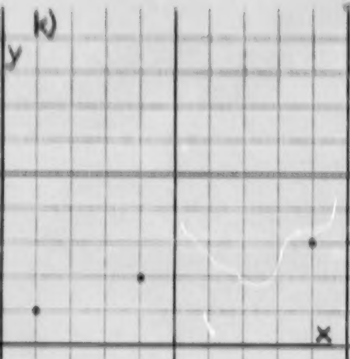
3.



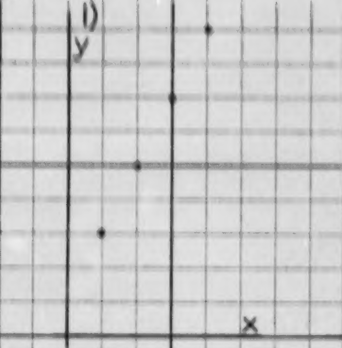
3. j)



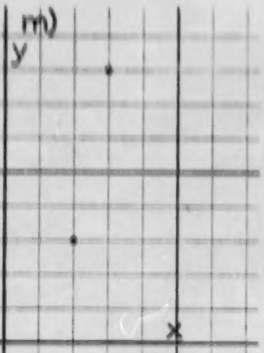
k)



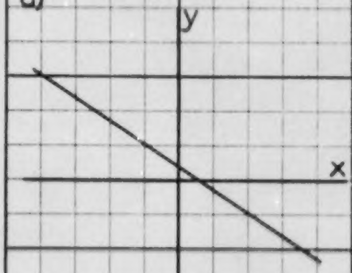
l)



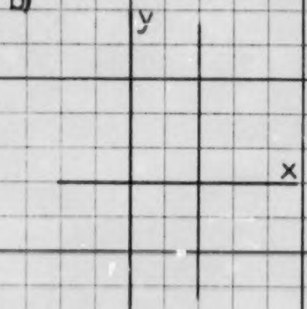
m)



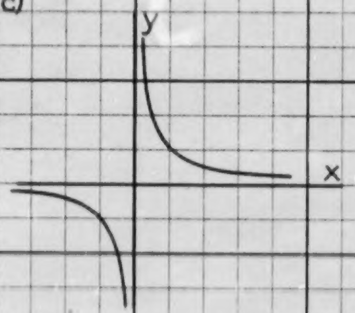
4. a)



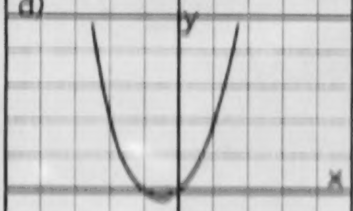
b)



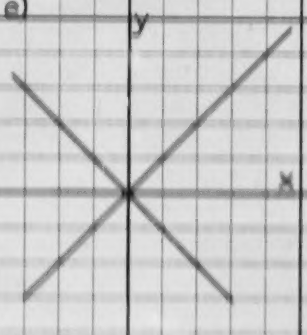
c)



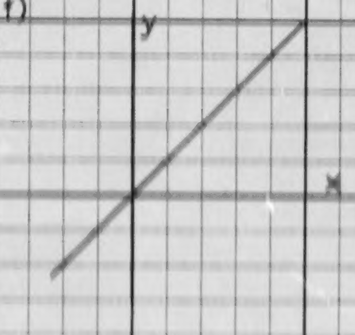
d)



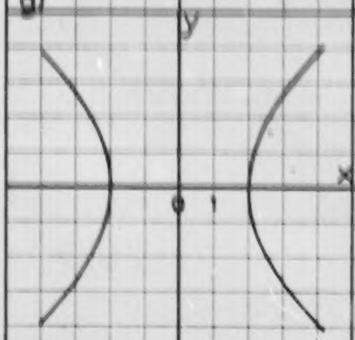
e)



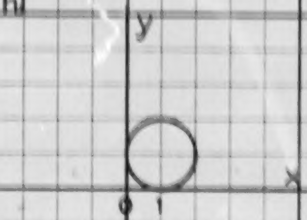
f)



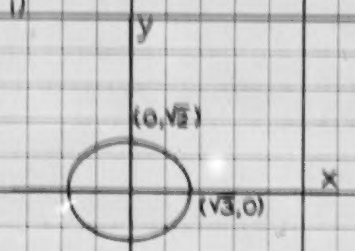
g)



h)



i)



5. Yes. See definition of a Cartesian Product.
6. No. The relation $\{(1,4), (2,9)\}$ where $U$ is the set of natural numbers less than 10 has $A = \{1,2\}$ and $B = \{4,9\}$. The Cartesian Product $A \times B = \{(1,4),(1,9),(2,4),(2,9)\}$. Hence, the subset of $U \times U$ exhibited as a relation is not itself a Cartesian Product but a subset of a Cartesian Product.
7. $y = x$ and $y = x^3$.
8. No.

## 1-14. Functions

All of the material we have studied so far leads up to the concept of a function. Since the subject to be covered during this semester is that of elementary functions, it is worthwhile spending some time clarifying the concept.

You will notice that we have taken two viewpoints with respect to functions. In one case, a function is looked upon as doing something, that is, associating with or assigning to the elements of one set those of another. In the other use, a function is looked upon as being something, a set of ordered pairs. If we take the ordered pair viewpoint, then a function becomes simply a special kind (many to one or one to one) of relation. Many mathematicians commonly think of a function in the first sense, and we have placed the major emphasis on this viewpoint. Of course no mathematician would always feel compelled to think in this way and in practice either viewpoint may be adopted in a given discussion since they can be shown to be logically equivalent. If we are discussing functions of functions (compositions) the association or mapping concept seems more convenient, while the ordered pair description is clearly preferable when representing a function as a graph. The student should become familiar with both.

You should be very careful at this stage to insist upon the proper use of functional notation and how to read it. If we write

$$f:x \longrightarrow y$$

then we read,

"The function $f$ which takes (or maps) $x$ into $y$."

If we write $y = f(x)$, we read

"$y$ is the value of $f$ at $x$."

$$f:R \longrightarrow R$$

is read

"$f$ is a function from the real numbers to the real numbers,"

or

"$f$ is a function from the reals to the reals."

The student should not be permitted to say that $y = f(x)$ is a function. This is a common error of usage. Many mathematicians still use $y = f(x)$ elliptically, but being mathematicians, they understand what they are doing. High school students, however, are apt to be very confused by this and we wish to do everything we can to be clear about the matter. Thus, $y = 3 - x$ is not a "linear function" although it may be used to define one over the reals, i.e.

$f:x \longrightarrow 3 - x$.

It is useful to visualize a function as a machine. The machine may be represented as a box with a hopper for input and a spout for output as in Figure TC, 1-14. The domain of f is then the input and the range is the output.

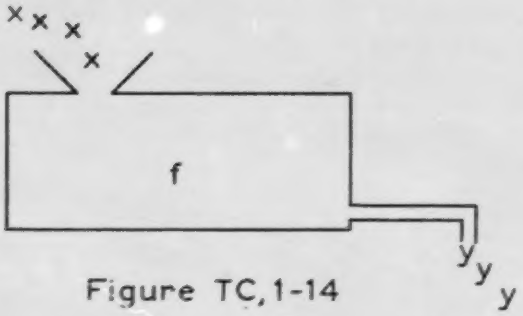


Figure TC, 1-14

If we place something in the hopper which is not in the domain, we can picture the machine jamming. (This is what actually happens when such an event occurs in an electronic computer; it stops.) This is a particularly useful device in handling the composition of functions, which we deal with in Chapter 3.

In explaining the function concept you will probably wish to make use of a variety of techniques. The machine is one such. Another approach might be to suggest a function from a domain consisting of the students in the class to a range consisting of the seats in the class and then ask for restrictions on the assignment so that it represents a function. For instance, two different seats could not be assigned to the same student; at least one seat would have to be assigned to each student, etc. Or again, inquire into the possibility of defining a function from the set of students to the set of their weights. Such examples are easy to devise and provide a means of focusing attention on the essential properties of a function. Also it is useful for the student to be aware of the fact that the domain and range of a function need not be numerical. Many times it is useful to consider a function from the reals, say, to a set of points in a plane or vice versa (Chapter 5).

It is also helpful to use examples from the sciences. You might ask the students what physicists mean when they say that the length of a metal bar is a function of the temperature of the bar, or that the pressure of a gas at a given temperature is a function of the volume it occupies. Make sure in each case that the students arrive at a function from the real numbers to the real numbers.

When introducing the idea of the function as a mapping, you should point out that there cannot be more than one arrow from each element of the domain while there can be any number of arrows to each element of the range. If there is just one arrow to each element in the range, then the function is said to be one-to-one and, as will be seen later, has an inverse.

The concepts of domain and range should be emphasized. It should be made clear that in order to define a function, we must have a domain. (It will prove valuable to the students if from time to time after you have completed the unit you stop and ask for the domain and range of whatever function you may be considering at the time.) We use the terminology "real function" so that the domain and range are specified as the set of real numbers.

## Exercises 1-14.

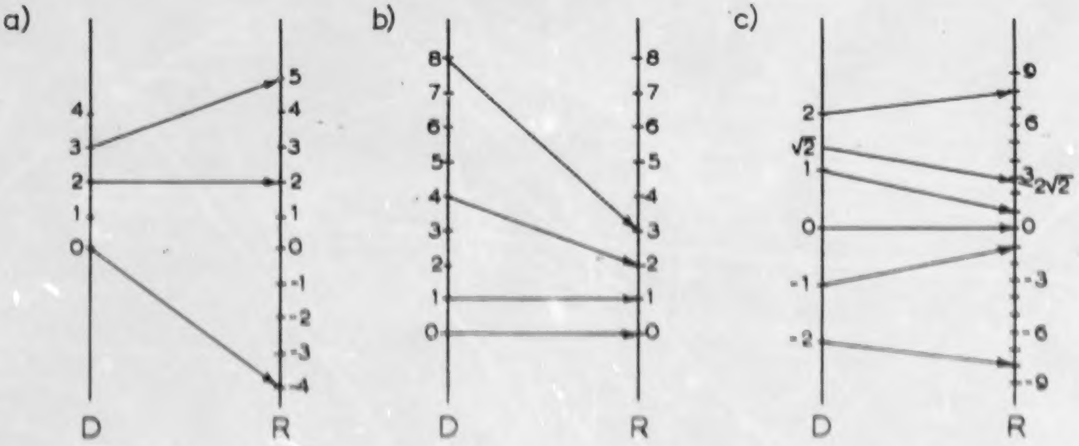
These are exercises in the use of functional notation. Most of the exercises involve familiar functions, though 3(e) and (f) may need some attention. The idea here is to exclude certain values from the domain and range. The question of how to find the excluded values in the range of a function has not been discussed in the text. One way is to sketch the graph of the function, not always easy. Another useful procedure, when feasible, is to solve the equation $y = f(x)$ for $x$ and to observe which values of $y$ do not define a value of $x$. Exercise 6 emphasizes again the unambiguous meaning of $\sqrt{\phantom{x}}$. Exercise 9 is an introduction to the idea of a point discontinuity, and the student should be made to see that there is a "hole" in the function at $x = 0$. The ordered pair $(0,0)$ in the graph of $f: x \longrightarrow x$ is not in the graph of $f: x \longrightarrow \frac{x^2}{x}$.

## Answers to Exercises 1-14.

1. e, because it is multiple-valued.

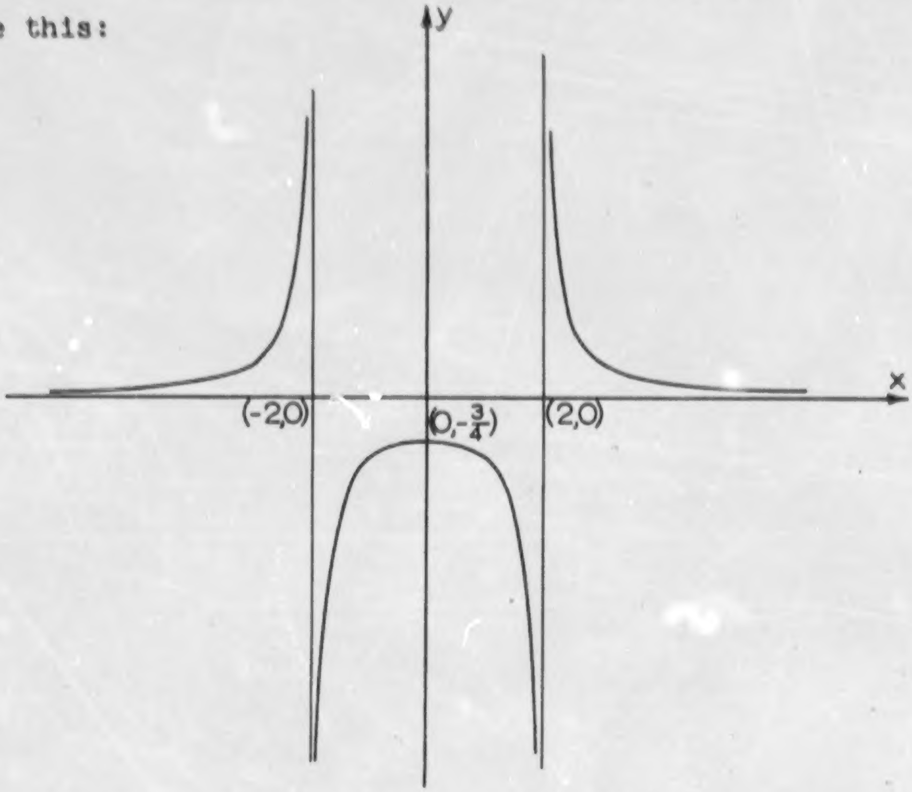
   ( $\sqrt{x}$ means only the positive square root.)

2.



| 3. | Domain | Range |
|---|---|---|
| a) | R | R |
| b) | R | Non-negative R |
| c) | Non-negative R | " " " |
| d) | R | " " " |
| e) | R except $x = 1$ | R except 1 |
| f) | R except $x = 2$ or $- 2$ | R except $- \frac{3}{4} < f(x) < 0$. |

Do not discuss f at length. Simply show graph to look like this:



If you test to find values of x for $\frac{3}{x^2 - 4}$ equal to numbers between $- \frac{3}{4}$ and 0, you will obtain imaginary values.

4. a) $f(0) = 1$ b) $f(-1) = -1$

c) $f(100) = 201$ d) $f(\frac{3}{2}) = 4$

5. a) $f(0) = 3$ b) $f(-1) = 6$

c) $f(a) = a^2 - 2a + 3$ d) $f(x - 1) = x^2 - 4x + 6$

6. a) $f(4) = 0$ b) $f(-5) = +3$

c) $f(5) = +3$ d) $f(a) = \sqrt{a^2 - 16}$

e) $f(a - 1) = \sqrt{a^2 - 2a - 15}$ f) $f(\pi) = \sqrt{\pi^2 - 16}$

7. a, b, c.

d is not the solution set of a function since it pairs more than one element of the range with each element of the domain.

8. $\{(-2, 0), (-1, \sqrt{3}), (0, 2), (1, \sqrt{3}), (2, 0)\}$

Domain $= \{-2, -1, 0, 1, 2\}$

Range $= \{0, \sqrt{3}, 2\}$

9. They are not the same function, since g does not include zero in the domain or range.

10. a) 4, - 4 b) 8 c) 16, - 16 d) 12, - 12

## 1-15. Graph of a function

The graph is perhaps the clearest means of displaying a function, since the story is all there at once. The student can observe the behavior of f for the various portions of the domain, and, in most cases, irregularities are obvious immediately. The difficulty is, of course, that some functions cannot be graphed, as, for example,

$$f:x \longrightarrow \begin{cases} 1 \text{ if } x \text{ rational,} \\ 0 \text{ if } x \text{ irrational} \end{cases}$$

Since high school students are not normally exposed to such functions, however, this is not a very serious obstacle.

The graph might best be introduced by using some function whose behavior is not too obvious. If the class has covered Section 1-12 (Locus), the students should be familiar with the graph of $|x| + |y| = 3$, as in Figure TC, 1-15a, and may then be interested in the graph of

$$F = \{(x,y) : |x + y| + |x - y| = 4\}$$

as shown in Figure TC, 1-15b.

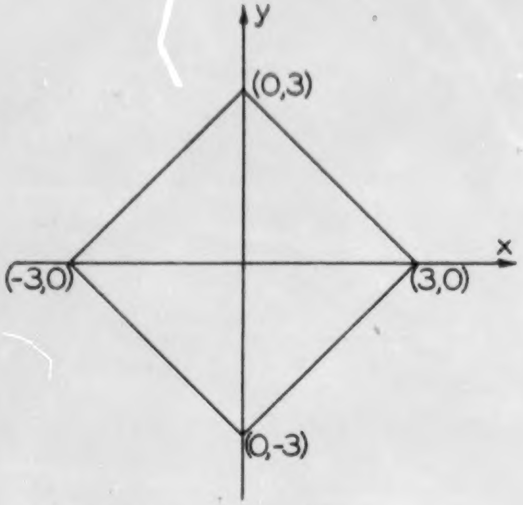


Figure TC,1-15a

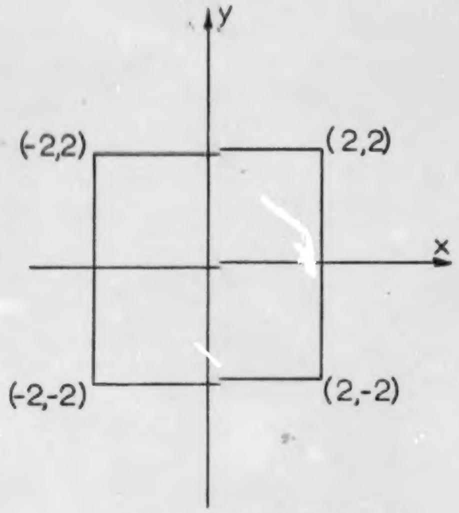


Figure TC,1-15b

Or again, you may wish to use the "greatest integer contained in" function, which, while not in the text, is easily explained and leads to some interesting configurations. We define

$$f:x \longrightarrow [x]$$

as the function which maps x into the greatest integer contained in x. Thus

$$f(1) = 1,\ f(3/2) = 1,\ f(1/2) = 0,\ f(-\ 3/2) = -\ 2, \text{ etc.}$$

The graph of the equation $y = [x]$ is in Figure TC, 1-15c. There are a number of interesting combinations which can be formed with $[x]$.

Figures TC, 1-15d to TC, 15g illustrate four of them.

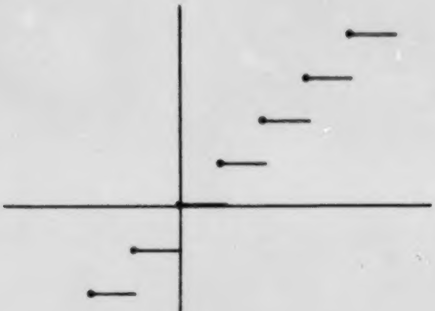


Fig. TC,1-15c

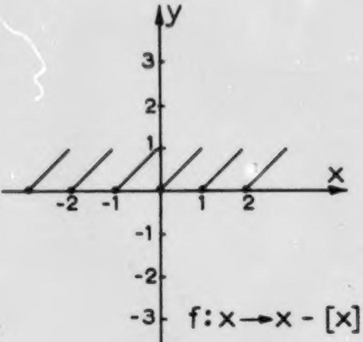


Fig. TC, 1-15d

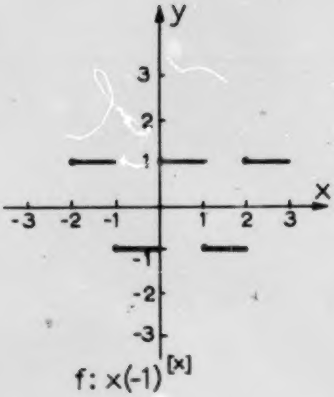


Fig.TC,1-15e

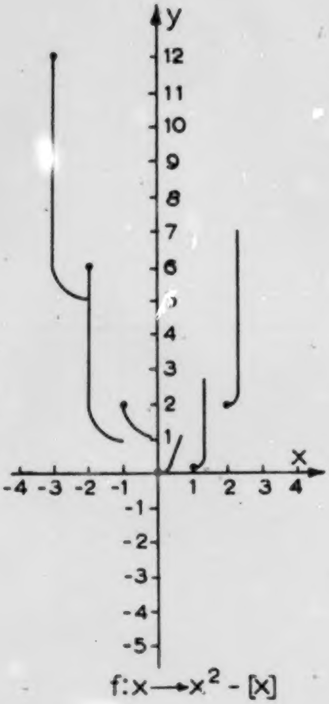


Fig.TC,1-15f

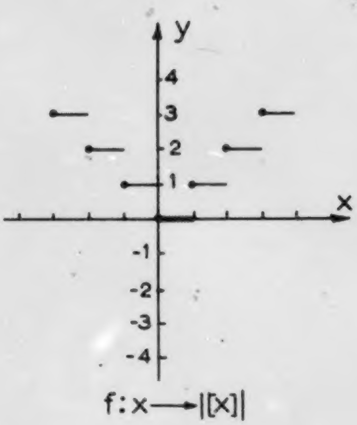


Fig.TC, 1-15g

Furthermore, an infinite checkerboard pattern such as that mentioned in this manual under Exercises 1-12 is given by

$$\{(x,y):[x] \text{ is even and } [y] \text{ is even}\}.$$

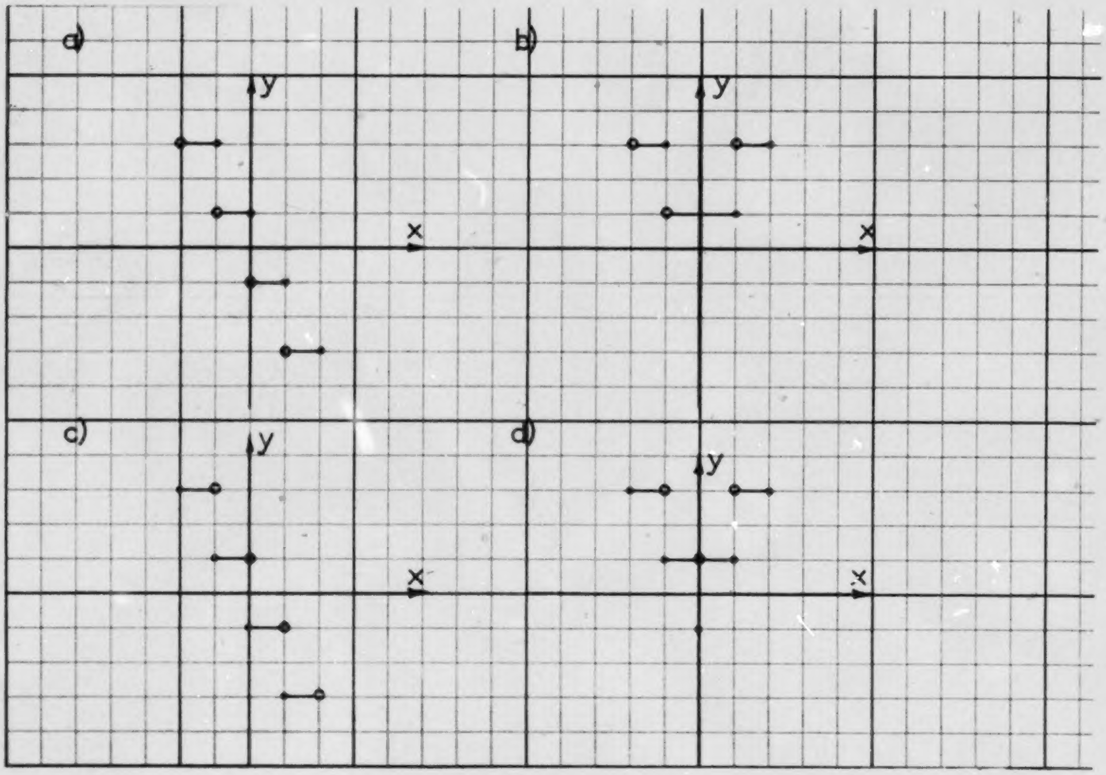
You may also find it helpful to sketch on the blackboard some figures similar to those in Figure 1-15f and have the students determine whether or not they represent functions by applying the vertical line test. Exercise 2 on page 68 is also a useful type of blackboard exercise and you will probably find it helpful to do one for them as an illustration before the students attempt to do exercise 2 themselves.

Answers to Exercises 1-15.

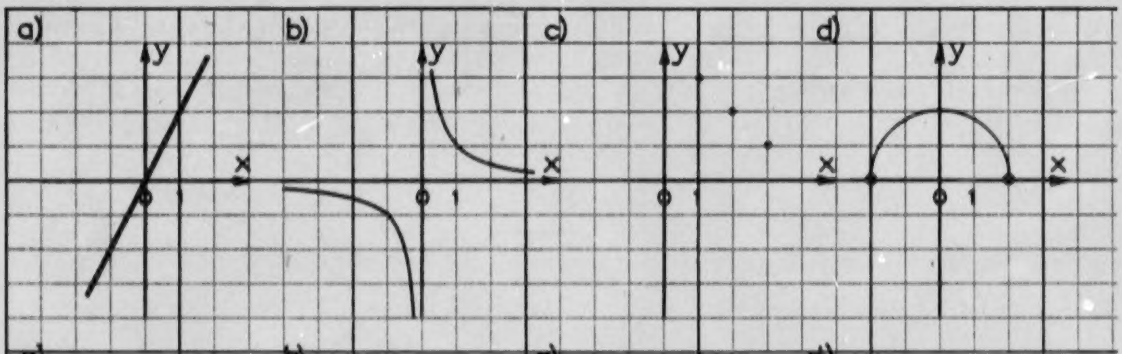
1. a and b are graphs of functions.

   c and d fail to meet the "vertical line test."

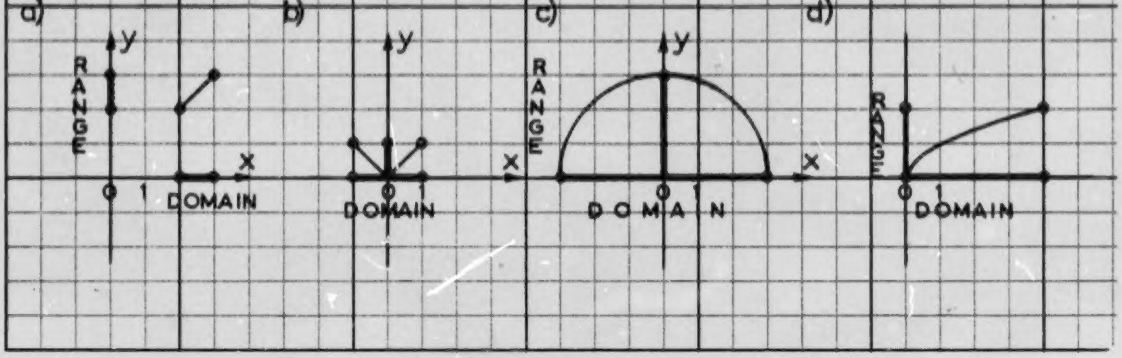
2.



3.



4.



## 1-16. Summary of Chapter 1

In this section we have presented a brief resume of the various concepts developed in the chapter for review purposes.

## Miscellaneous Exercises

The problems contained in this set are in general more difficult than the problems contained in the body of the chapter. They are designed to test a student's ability to use the concepts of the chapter in somewhat original situations. Careful reading and interpretation of the symbolism is the key to success, assuming that the basic ideas have been mastered.

If you feel pressed for time, it is not necessary that any of these problems be assigned. On the other hand, you might assign some or all of them as a project to be carried on over an extended period of time. It would also be possible to use them as review exercises near the end of the course by assigning one problem each night along with the current assignment.

Answers to Exercises for Chapter 1.

1. a) $A = \{-3, -2, -1, 0, 1, 2, 3\}$

   b) $B = \{-3, -2, -1, 1, 2, 3\}$

   c) $C = \{0, 1, 4, 9, 16\}$

   d) $D = \{0, 1, 2, 3, 4, 5\}$

2. a) $A \times B = \{(-1,1), (-1,5), (-1,7), (0,1), (0,5), (0,7), (3,1), (3,5), (3,7), (4,1), (4,5), (4,7)\}$

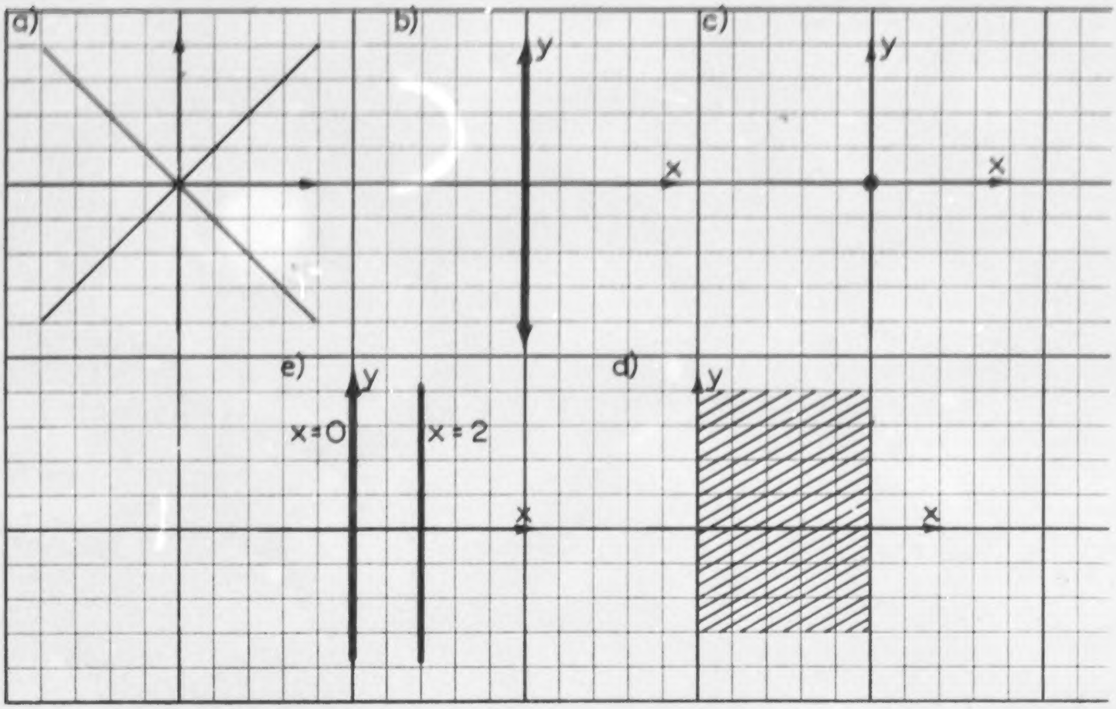
   b) $= \{0, 1, 4, 5, 6, 7, 8, 9, 10, 11\}$

   c) $= \{-7, -5, -1, 0, 3, 4, 15, 20, 21, 28\}$

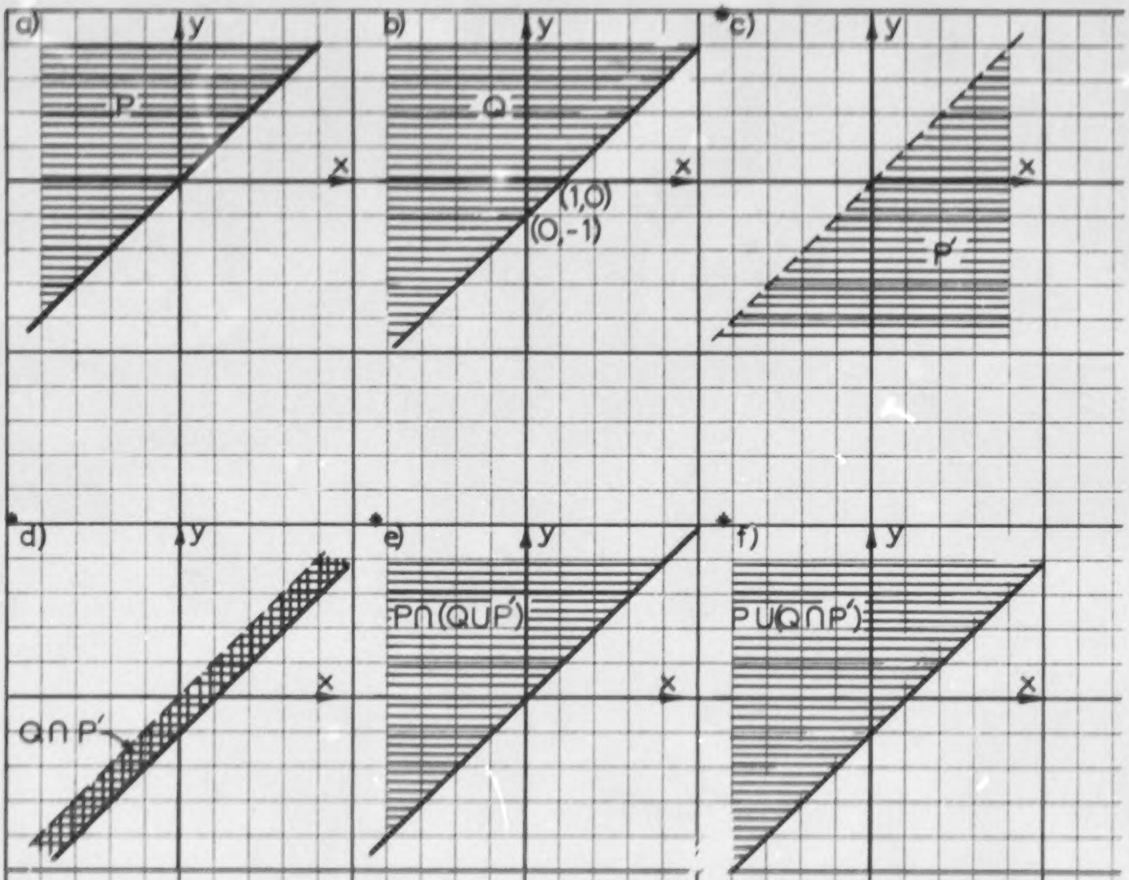
   *d) $A \cup B = \{-1, 0, 1, 3, 4, 5, 7\}$

   *e) $A \cap B = \emptyset$

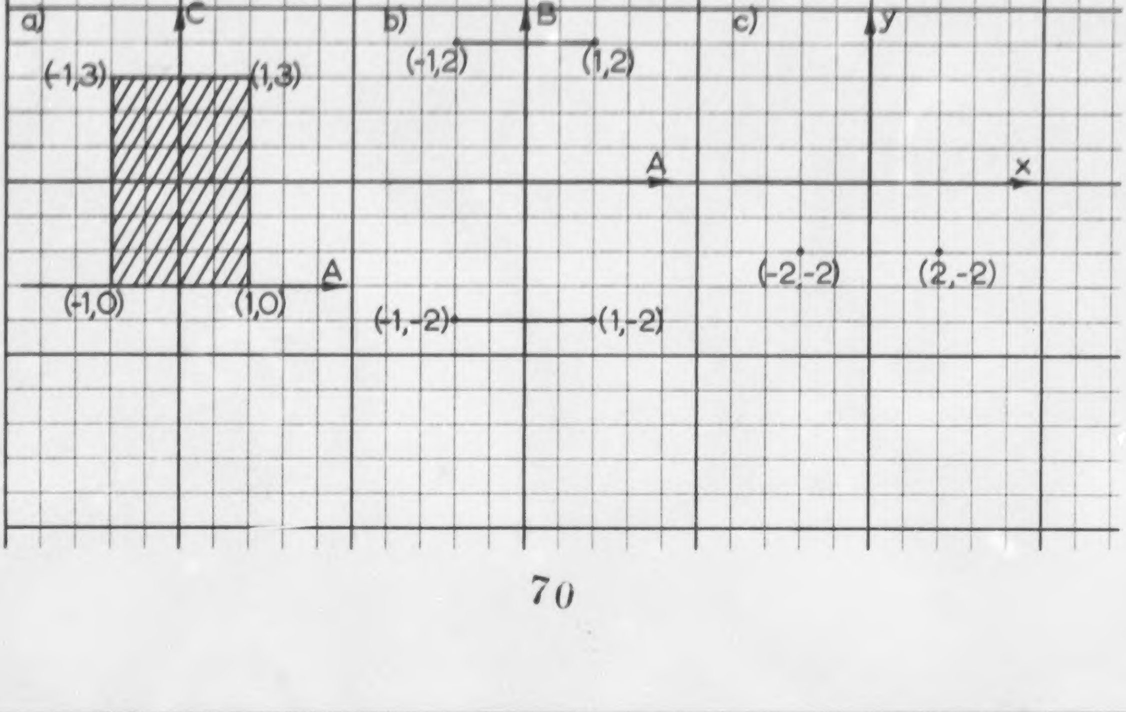
3.



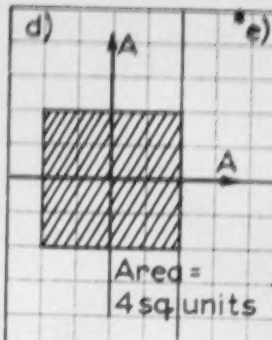
4.


a)
y
x
P
b)
y
x
Q
(1,0)
(0,-1)
c)
y
x
P′
d)
y
x
Q∩P′
e)
y
x
P∩(Q∪P′)
f)
y
x
P∪(Q∩P′)

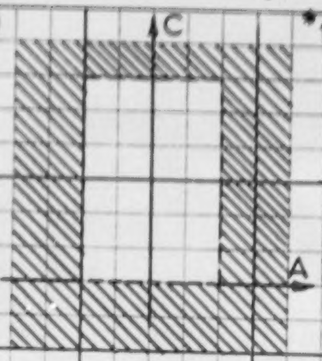
5.


a)
C
A
(-1,3)
(1,3)
(-1,0)
(1,0)
b)
B
A
(-1,2)
(1,2)
(-1,-2)
(1,-2)
c)
y
x
(-2,-2)
(2,-2)

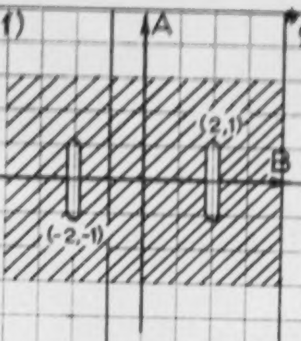
5. d)


A
A
Area =
4 sq units

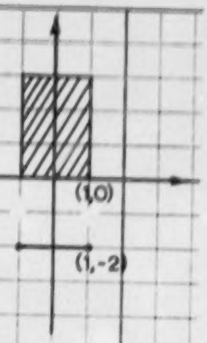
*e)


C
A

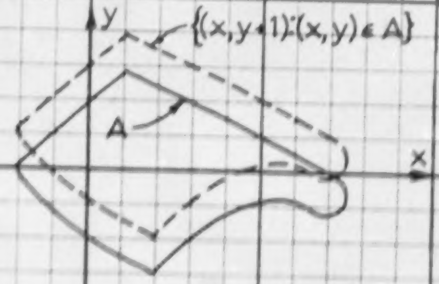
*f)


A
(2,1)
B
(-2,-1)

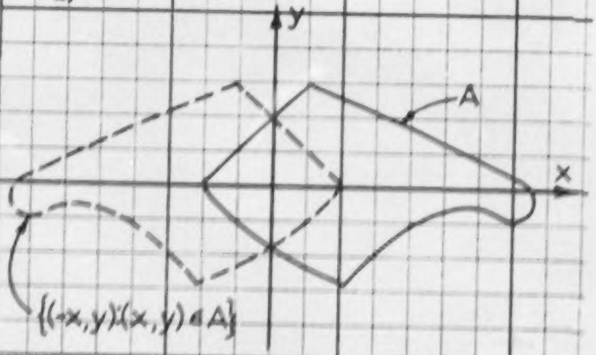
*g)


(1,0)
(1,-2)

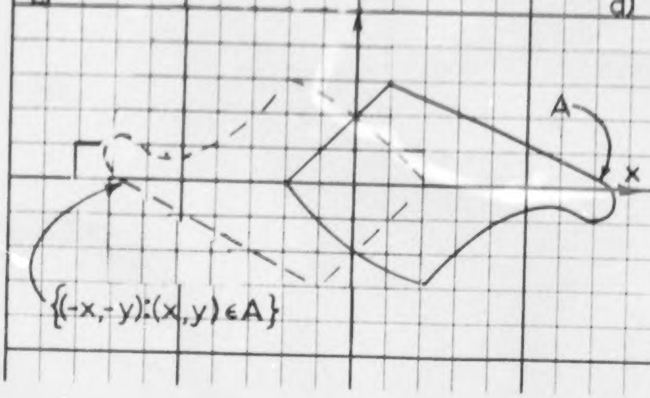
6. a)


y
{(x,y+1):(x,y) ∈ A}
A
x

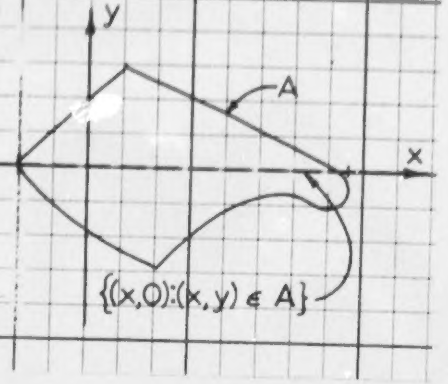
b)


y
A
x
{(-x,y):(x,y) ∈ A}

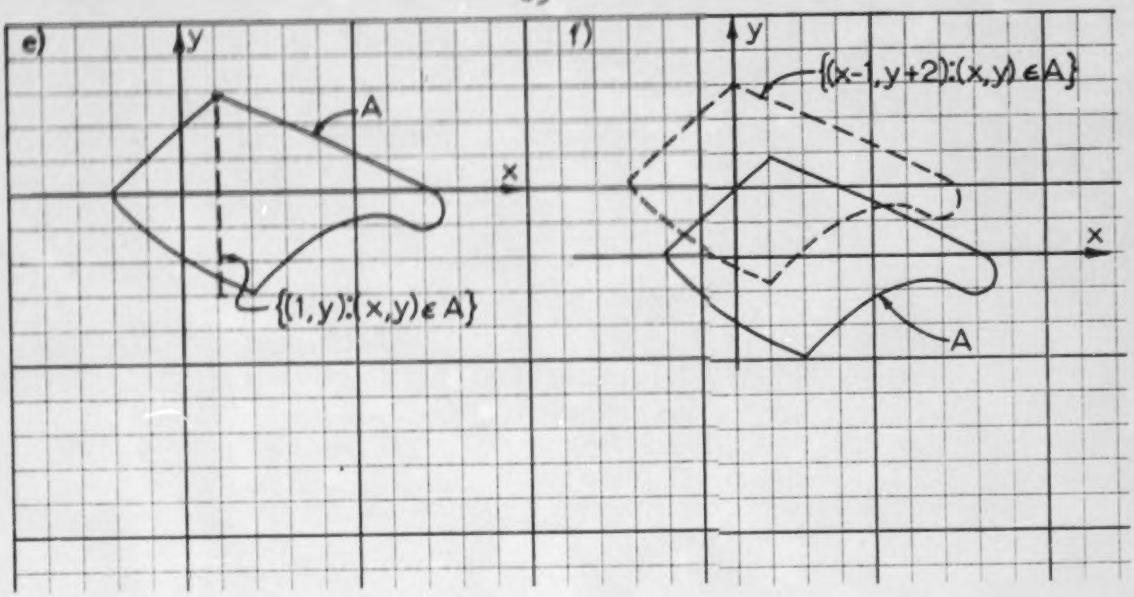
c)


A
x
{(-x,-y):(x,y) ∈ A}

d)


y
A
x
{(x,0):(x,y) ∈ A}

6.



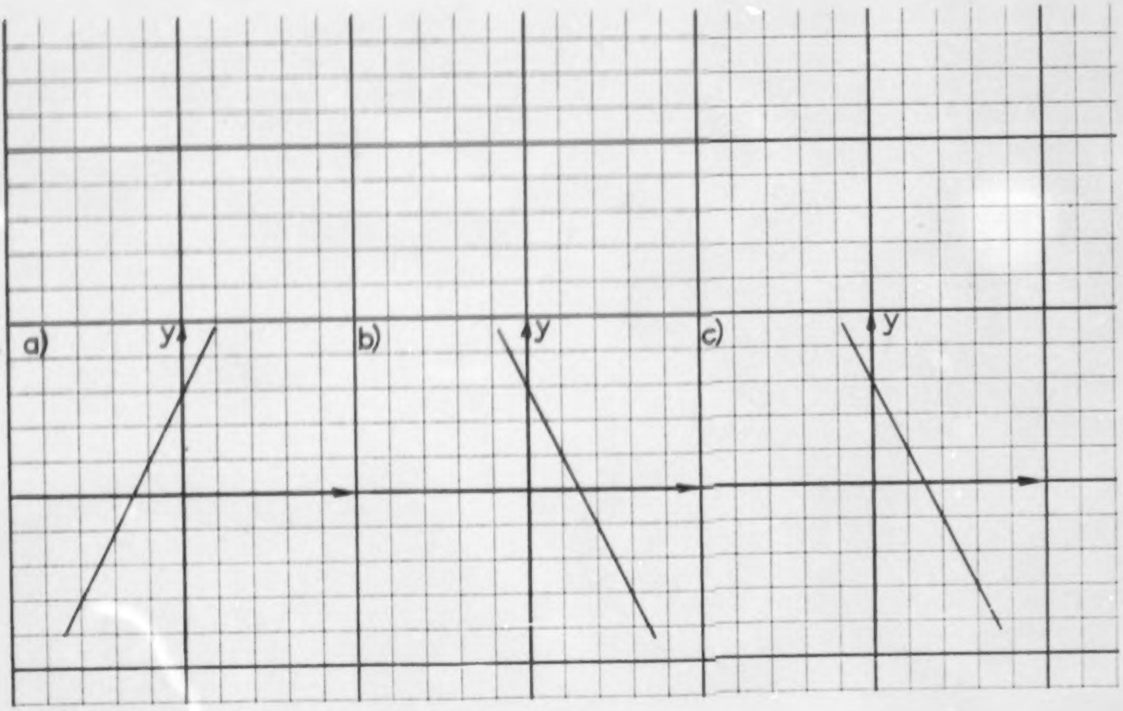
7. a) A = {(0,- 2)}

b) Note: w(s + 2) = (S + 2)(S - 1). B = {4,18,40}

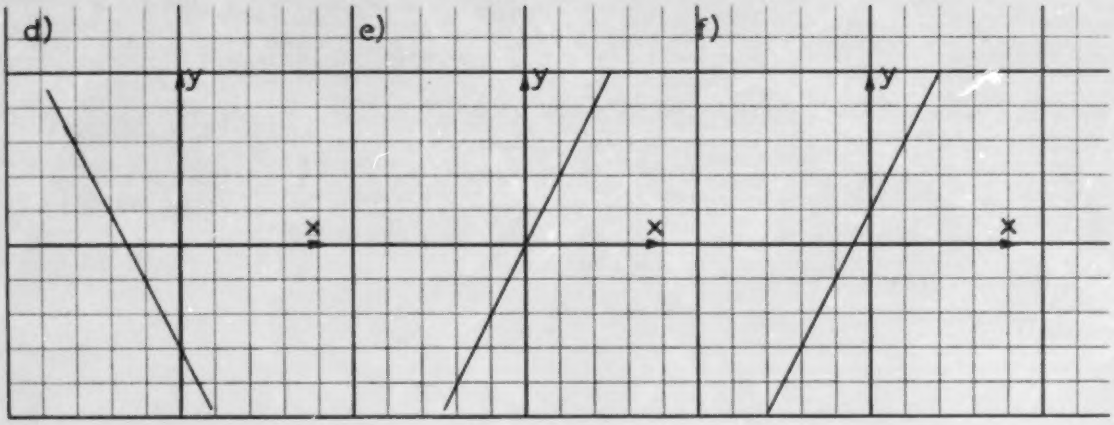
c) C = {(1,- 2)(3,0)(5,10)(7,28)}

d) D = {(- 1,0),(- 1,- 2),(0,0),(0,- 2),
(1,0),(1,- 2), (2,0),(2,- 2)}

8.



8.



In drawing these graphs we are plotting points (x,y) to meet the conditions of the problem.

9. a) $p(1) = 0$

b) $q(3) = -7$

c) $q(1) + p(-2) = -1 + 3 = 2$

d) $p(4 + q(0)) = p(4 + 2) = 35$

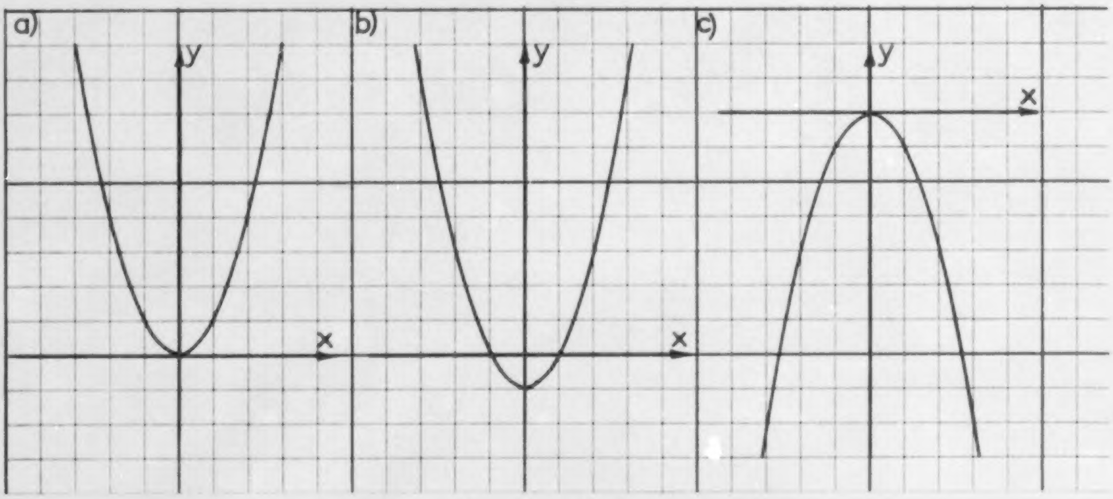
e) $q(p(q(2))) = q(p(-4)) = q(15) = -43$

f) $p(1 + p(1 + p(1)) = p(1 + p(1 + 0)) = p(1 + 0) = 0$

g) $p(q(x)) = 4 - 12x + 9x^2 - 1 = 3(1 - 4x + 3x^2)$

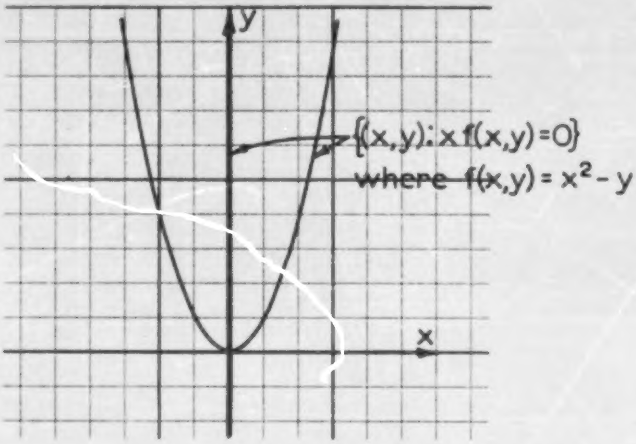
h) $q(p(x)) = 5 - 3x^2$

10.



d) is exactly the same as a)

e)



*11. If you reread the definition for $A \setminus B$ you will find it to agree with the definition for $A \cap B'$.

Of the four parts of this problem, the first can be shown very easily. The other 3 parts will be done in 3 different methods. It will be advisable to use the other two methods with each of the 3 parts.

a) $A \setminus A$ is the same as $A \cap A'$ (which has been shown in Exercise 1-7-2f) $= \emptyset$

b) Several parts of Exercise 1-7-6 will be used here in this proof:

$A \setminus (B \cup C) = A \cap (B \cup C)' = A \cap (B' \cap C') =$

$A \cap A \cap (B' \cap C') = (A \cap B') \cap (A \cap C') =$

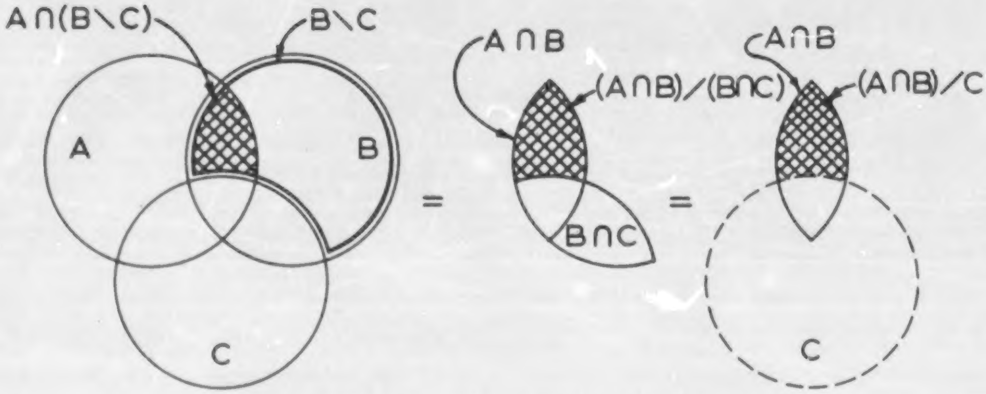
$(A \setminus B) \cap (A \setminus C)$

c) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

This demonstration will consist of an examination of the conditions described on the left side of the equation and then on the right side. $A \setminus (B \cap C)$ consists of those elements in $A$ which are not in both $B$ and $C$. If we apply this set of conditions to $(A \setminus B) \cup (A \setminus C)$ we find the same element either in $(A \setminus B)$ or $(A \setminus C)$ or both. But it definitely appears in the union of these two sets. Elements not in the set on the left side would either not be in $A$ or be in $B$ and $C$. This same condition will rule it out of the right side.

d) $A \cap (B \setminus C) = (A \cap B) \setminus (B \cap C) = (A \cap B) \setminus C$

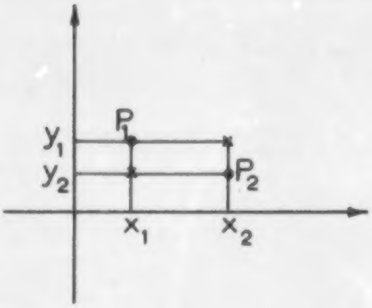
We shall use a Venn diagram method to prove this



SHADED AREAS ARE EQUAL

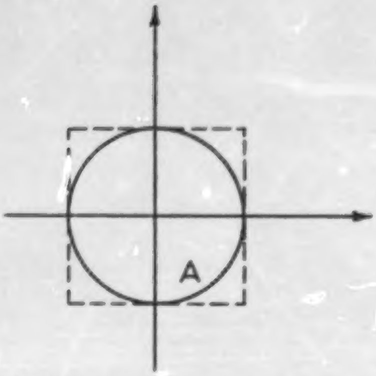
*12. Any set $A$ of two or more points whose projections on the x-axis or y-axis would not coincide, would lead to an inclusion not an equality. For examples:

a)



If $A = \{P_1, P_2\}$ then $p(x,y) = \{x_1, x_2\}$ and $q(x,y) = \{y_1, y_2\}$. Then $\{(p(x,y), q(x,y)) : (x,y) \in A\} = \{(x_1,y_1), (x_2,y_2), (x_1,y_2), (x_2,y_1)\}$ and $A$ is a Proper subset of this set.

b)



The set of points in the circle A would produce as a Cartesian Product set the ----- square which would enclose it.

Basically, this gives a symbolic method of filling out a minimum rectangle which would contain the area indicated. (It would be the "minimum" rectangle whose sides are $\perp$ or $\|$ to the coordinate axes.

*13. a) Prove: $A \cup B = \{x : f(x) - g(x) = 0$

Given: $A = \{x : f(x) = 0\}$ and $B = \{x : g(x) = 0\}$

Proof: If $x_a \in A$ then $f(x_a) \cdot g(x_a) = 0 \cdot g(x_a) = 0$, so all $x_a$($\in A \cup B$ by definition of union) will be $\in \{x : f(x) \cdot g(x) = 0\}$ by the first line. Similarly, for $x_b \in B$. $\therefore A \cup B \subset \{x : f(x) \cdot g(x) = 0\}$

If $x_c \in \{x : f(x) \cdot g(x) = 0\}$, Then $f(x_c) \cdot g(x_c) = 0$. And since these are functions from R to R for which no divisors of zero are known to exist, either $f(x_c) = 0$ or $g(x_c) = 0$. If the first is true $x_c \in A$, if the second $x_c \in B$ by definition. So $\{x : f(x) \cdot g(x) = 0\}$ $\subset A \cup B$ and so the equality is established.

Useful References

Report of the Commission on Mathematics - Appendices, College Entrance Examination Board, 1959, Chapters 1. 2, 9.

The Growth of Mathematical Ideas, Grades K-12, 24th Yearbook, NCTM, 1959, Chapters 3, 8.

Insights into Modern Mathematics, 23rd Yearbook, NCTM, 1957, Chapter 3.

Elements of Modern Mathematics, K. O. May, Addison-Wesley Publishing Co., Reading, Mass., 1959.

Fundamentals of Freshman Mathematics, C. B. Allendoerfer and C. O. Oakley, McGraw-Hill Book Co., New York, 1959, Chapters 6, 8, 9.

## Illustrative Test Questions for Chapter 1

The following twenty questions (with answers appended) have been included in order to assist you in the preparation of tests and quizzes. The order of the items is approximately the same as the order in which the various concepts being tested appear in the text. This means that you can use selected problems from this list before the chapter has been completed. The starred items deal with ideas contained in the optional material of Sections 1-6, 7, and 12. Problem 2 is included for the benefit of those who stress proofs in the course, and it would not be a fair question for students who have not seen a proof of this kind before.

For a short quiz one or two problems from this list would be sufficient; a full period (40 to 50 minutes) test might contain anywhere from five to ten of the problems. It would be a mistake to give all twenty questions as a chapter test unless at least two class periods were planned for it.

1. Represent the following sets symbolically:
    a) the prime numbers of the form $4n - 1$ less than 30 (tabulate the elements);
    b) the roots of the equation $x^5 + x^2 + 2 = 0$ (use the set-builder notation).
2. If A, B, and C are sets such that $A = B$ and $B = C$, prove that $A = C$.
3. Given $A = \{1, 3\}$ and $B = \{1, 2, 3, 4\}$, find all possible sets C such that $A \subset C \subset B$.

*4. Given $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$, and $B = \{1, 2, 4\}$, tabulate

a) $A' \cup B'$;

b) $A \cap B$;

c) $(A \cap B') \cup (A' \cap B)$.

5. Graph the solution set of $|x + 1| < 3$, where the universe is the set of integers.

*6. If $A = \{x : 2x + 1 > 7\}$, $B = \{x : x + 3 < 0\}$, and $U = \{-4, -2, 0, 2, 4, 6\}$, tabulate

a) $A' \cap B'$

b) $(A \cup B) \cap B'$.

7. Find $u$ and $v$ such that $(2u + 1, v - 1) = (u - 1, 3v)$.

8. If $U = R$, sketch the graph of $S = \{(x, y) : y < x$ and $1 < x < 2\}$.

9. Sketch the graph of the Cartesian product $A \times B$, where $A = \{2, 3, 4, 5\}$ and $B = \{y : 1 < y < 3$ and $y \in R\}$.

10. Given $U = \{1, 2, 3, 4\}$ and the Cartesian product $U \times U$, list the members $(x, y)$ of $U \times U$ such that $x \geq y + 1$.

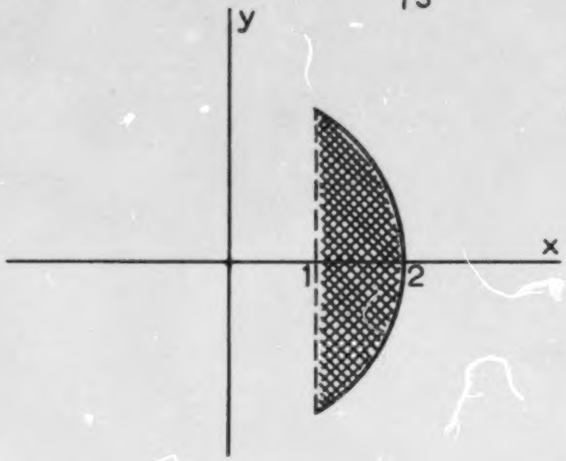
*11. Graph the equation $|x| + 2|y| = 4$.

*12. Given $U = R$, $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$, and $B = \{(x, y) : -1 \leq x \leq 3\}$, graph the locus of

a) $A \cap B$;

b) $A \cup B'$.

*13. Find a set selector whose set has the locus below: (The dotted line is not part of the locus and the arc shown has a radius of 2.)



14. Express the following relations in set-builder notation, regarding the universe as the set of natural numbers less than 10:

    a) $\{(4, 2), (6, 3), (8, 4)\}$;

    b) $\{(7, 9), (8, 8), (8, 9), (9, 7), (9, 8), (9, 9)\}$.

15. The following relations may be expressed in the form A × B. Find A and B.

    a) $\{(1, 2), (1, 3), (3, 2), (3,3), (7, 2), (7, 3)\}$;

    b) $\{(x, y) : y = 3\}$, $U = R$;

    c) $\{(x, y) : x > 3 \text{ and } 0 \leq y \leq 4\}$,

    $U = \{(x, y) : x \leq 5, y \leq 5, \text{ and } x, y \in R\}$.

16. For each of the following functions, specify the domain where the range is given and the range where the domain is given:

    a) $f: x \longrightarrow y = \sqrt{x - 1}, y \in R, y \geq 0$;

    b) $f: x \longrightarrow y = \frac{x + 2}{x + 1}, x \in R, x \neq -1$.

17. Given the function $f: x \longrightarrow f(x) = x^2 + 2, x \in R$, find

    a) $f(3)$;

    b) $f(6)$;

    c) $f(3/2)$;

d) Elements of the domain which have 18 as an image.

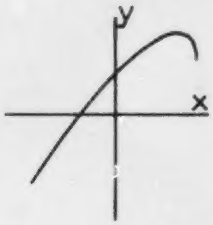
18. Indicate which of the following phrases apply to the graphs below:

a) a relation between $x$ and $y$;

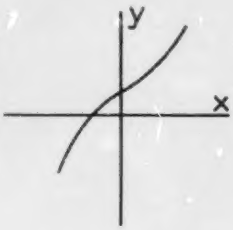
b) a function that takes $x$ into $y$;

c) a function that takes $y$ into $x$:

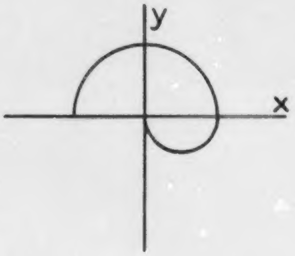
(1)



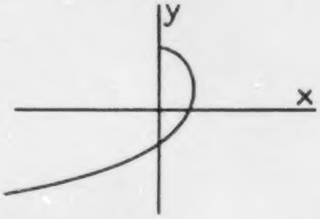
(2)



(3)



(4)



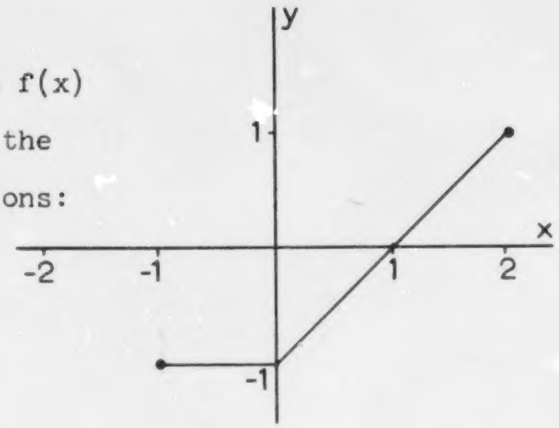
19. Given the function $f: x \longrightarrow f(x)$ graphed at the right, sketch the graph of the following functions:

a) $g: x \longrightarrow -f(x)$;

b) $h: x \longrightarrow -f(-x)$;

c) $k: x \longrightarrow f(|x|)$;

c) $m: x \longrightarrow |f(x)|$.



20. Graph the following functions in the universe of real numbers, indicating the domain and range on the appropriate axes:

a) $f: x \longrightarrow 1 + \sqrt{x}$, $x > 1$;

b) $F = \{(x, y) : y = 1 + |x - 1| \text{ and } -1 \leq x \leq 2\}$.

Answers to Illustrative Test Questions

1. a) $\{3, 7, 11, 19, 23\}$

   b) $\{x : x^5 + x^2 + 2 = 0\}$ (The universe in this case is the set of complex numbers, although it is not obvious by inspection.)

2. This problem seems so intuitively obvious that students may wonder why a proof is needed. The point is that we are using the = sign in a new way, and until the property exhibited in this exercise has been proved we can do no more than assume that it is true.

   The proof is to show that by definition every element of the set $A$ is an element of the set $C$, and that every element of $C$ is an element of $A$. Then by definition, $A = C$.

   One form of the proof follows:

   Given: $A$, $B$, and $C$ are sets such that $A = B$ and $B = C$.

   Prove: $A = C$.

   Since $A = B$, then $a \in A$ implies $a \in B$,
   and since $B = C$, then $a \in B$ implies $a \in C$;
   therefore, $a \in A$ implies $a \in C$.
   Also, since $B = C$, then $c \in C$ implies $c \in B$,
   and since $A = B$, then $c \in B$ implies $c \in A$;
   therefore, $c \in C$ implies $c \in A$.

   Hence, $A = C$, by definition.

3. $\{1, 3\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$

*4. a) $\{2, 3, 4, 5\}$

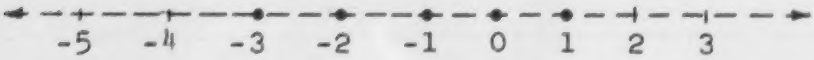
b) $\{1\}$

c) $\{2, 3, 4, 5\}$

5. Since the universe is the set of integers, this problem can be done by trial using direct substitution, or formally by manipulation. The solution set is

$$\{x : x \text{ is an integer and } -4 < x < 2\}$$

and the graph is:



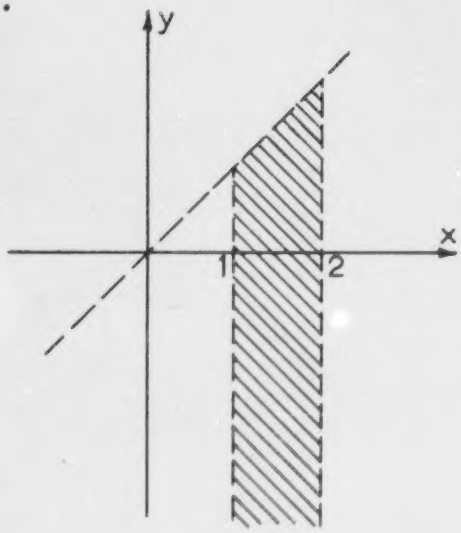
*6. $A = \{4, 6\}$, $B = \{-4\}$

a) $\{-2, 0\ 2\}$

b) $\{4, 6\}$

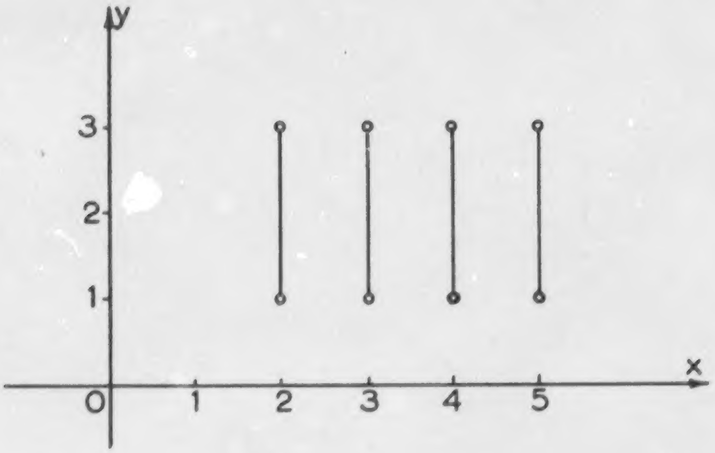
7. $u = -2$, $v = -1/2$

8.



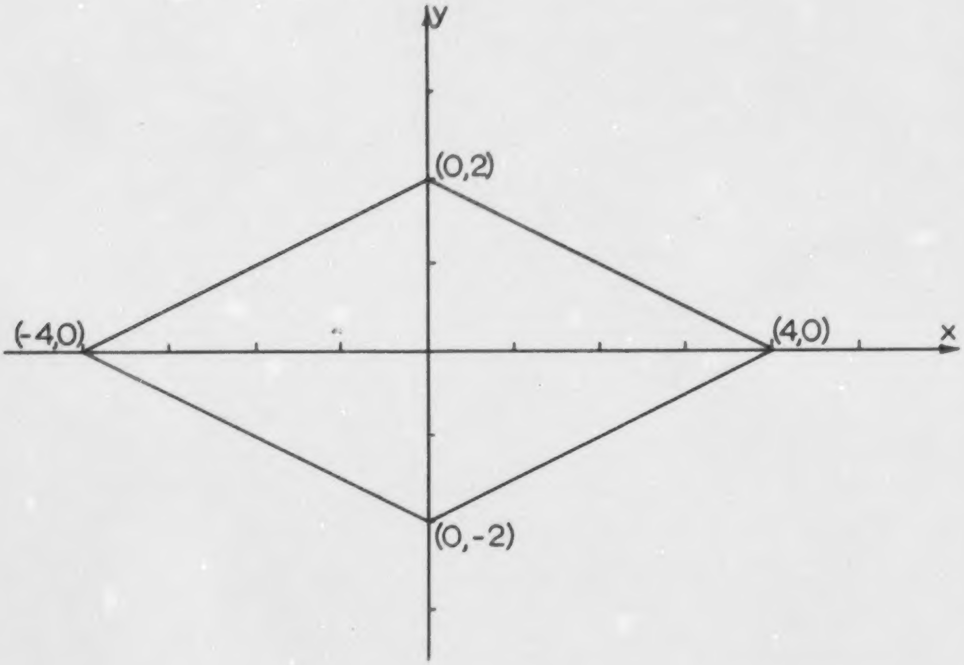
The shaded region is the graph of S. It extends indefinitely in the negative y direction. It does not include the lines $y = x$, $x = 1$, and $x = 2$.

9.

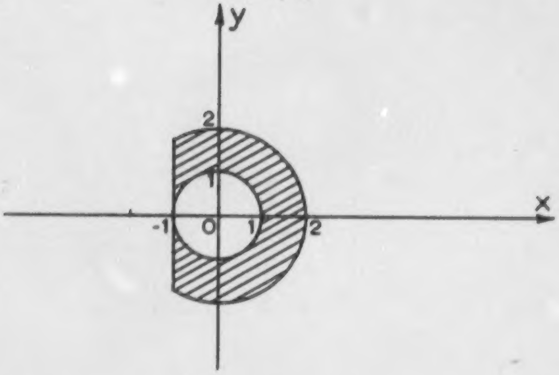


10. {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}

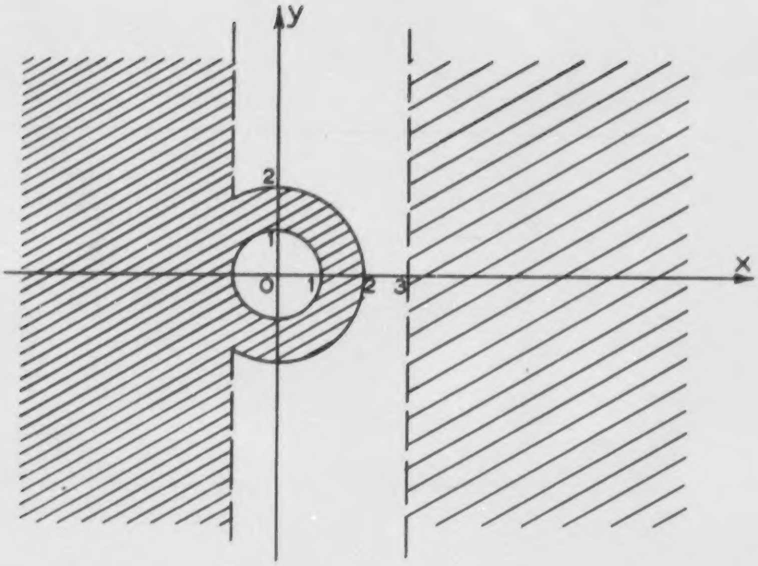
*11.



*12. a)



b)



*13. $\{(x,y) : x, y \in R,\ x^2 + y^2 \leq 4, \text{ and } x > 1\}$

or an equivalent form, such as

$U = R,\ A = \{(x,y) : x^2 + y^2 \leq 4\},\ B = \{(x,y) : x > 1\}$:

the required locus is the graph of $A \cap B$.

14. a) $\{(x,y) : y = x/2 \text{ and } x \geq 4\}$

or $\{(x,y) : x = 2y \text{ and } y \geq 2\}$.

b) $\{(x,y) : x + y \geq 16\}$.

15. a) $A = \{1, 3, 7\}$, $B = \{2, 3\}$

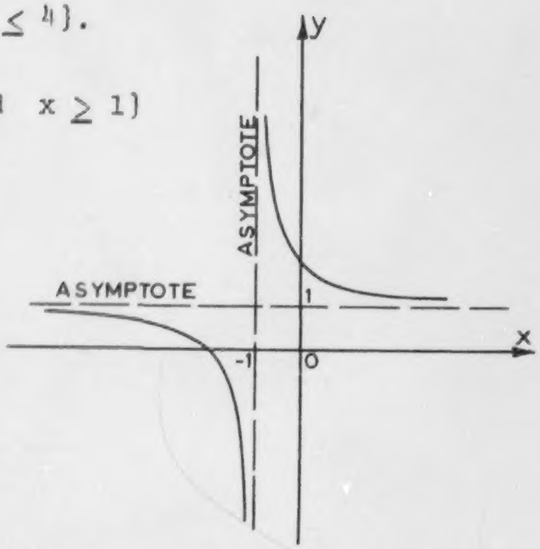
b) $A = R$, $B = \{3\}$

c) $A = \{x : x \in R \text{ and } 3 < x \leq 5\}$,

$B = \{y : y \in R \text{ and } 0 \leq y \leq 4\}$.

16. a) Domain $= \{x : x \in R \text{ and } x \geq 1\}$

b) The graph of $f$ is



From the graph we see that $y = 1$ is not contained in the range of $f$.

Solving $y = \frac{x + 2}{x + 1}$ for $x$ gives

$x = \frac{2 - y}{y - 1}$, for which we see that $x$ is undefined when $y = 1$.

Answer: Range $= \{y : y \in R \text{ and } y \neq 1\}$

17. a) 11

b) 38

c) 17/4

d) 4, -4

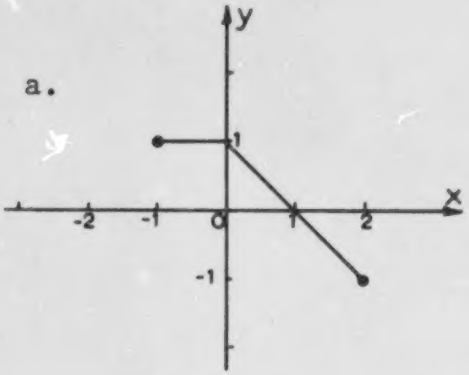
18. (1) a, b

(2) a, b, c

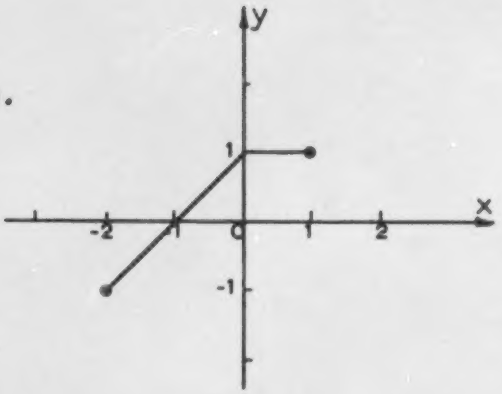
(3) a

(4) a, c

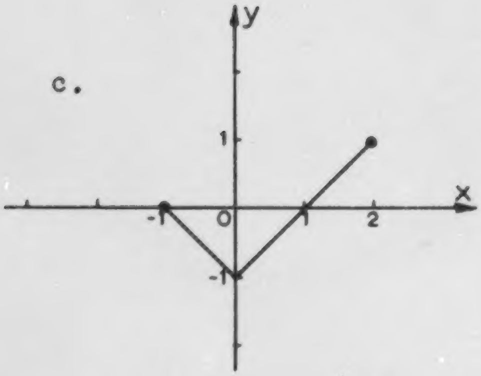
19. a.



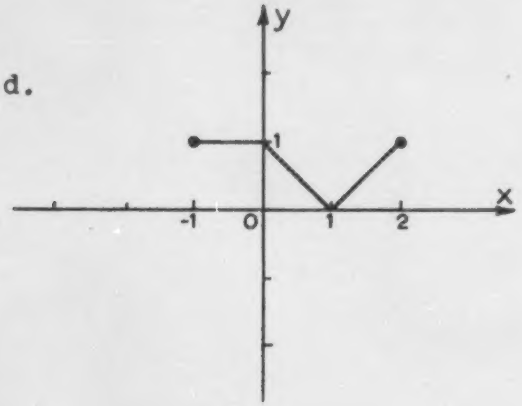
b.



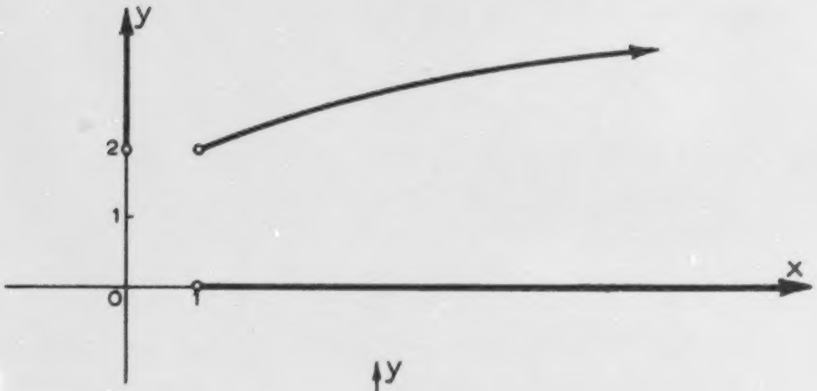
c.



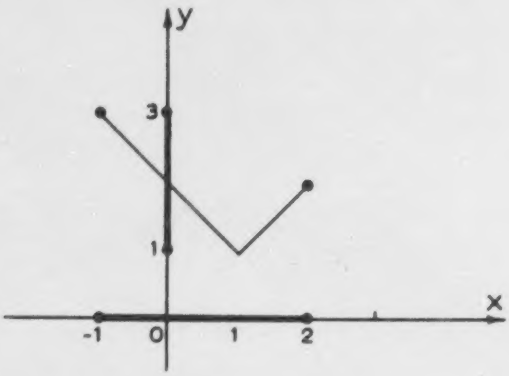
d.



20. a.



b.



## Chapter 2

## POLYNOMIAL FUNCTIONS

### Introduction.

This is the first of two chapters on polynomial functions. It is concerned with the fact that the graph of a polynomial is approximately straight sufficiently near any one of its points. The straight line which approximates the graph near a point P is called the tangent to the graph at P. In this chapter, elementary methods are developed for writing the equation of the tangent to the graph of $f: x \rightarrow f(x)$ first at $(0, f(0))$ and then at any point $(h, f(h))$. When the student has learned this technique he can locate maximum and minimum points on the graph.

In chapter 3 we shall be concerned with the zeros of a polynomial. It will help to locate these zeros approximately if the student can draw the graph of the polynomial. The location of maximum and minimum points should assist in this process.

The equations of the tangents to polynomial graphs are ordinarily found by calculus methods. Indeed the slope function of Section 2-5 is customarily called the derivative of the polynomial. We have deliberately avoided the use of this word and the $\frac{dy}{dx}$ notation. Teachers are urged not to give away the show by using calculus language. It is our belief that a short formalistic treatment of calculus can do more harm than good. It is felt that the present treatment will furnish the student with a good intuitive background for a course in calculus without giving him the impression that he already knows the subject.

Because we are concerned with approximating graphs by straight lines it seemed advisable to review briefly those functions whose graphs are straight lines. This is done in Section 2-1. Section 2-2 is divided by exercise lists into three lessons. All of the essential ideas of the chapter are developed for quadratic functions in which the problem of linear approximation occurs in its simplest possible form. In Sections 2-3 and 2-4 these ideas are applied to polynomials of higher degree and Section 2-5 summarizes the results in terms of the so-called slope function. In Section 2-6 the technique of finding the tangent is applied to the solution of maximum and minimum problems of considerable variety and interest. The amount of time which is available for these applications will probably vary considerably from school to school. The omission of this section altogether would not interfere with subsequent chapters. However, it is hoped that time will be available for a fair number of these problems, because of the interest which they are capable of arousing in the student.

## 2 - 1. Constant Functions and Linear Functions

Although the ideas of this section should be familiar to the student, it is unlikely that he will have encountered them in the language of mapping used in Chapter 1. This section will, therefore, at the same time give a review of important material and valuable practice in the use of functional notation.

## Answers to Exercises 2 - 1.

1. (a) Slope = 3
   (b) Slope = 2
   (c) Slope = $-\frac{1}{2}$
   (d) Slope = $\frac{4}{3}$

2. (a) $f:x \rightarrow -2x + 6$
   (b) $f:x = -2x - 7$
   (c) $f(x) = -2x + 7$
   (d) $f(x) = -2x + 13$

3. (a) Slope = $\frac{-3-4}{1-0} = -7$
   (b) Slope = $\frac{-3-3}{1-2} = 6$
   (c) Slope = $\frac{-3-5}{1-5} = 2$
   (d) Slope = $\frac{-3+13}{1-6} = -2$

4. (a) $f(x) = 3x - 2$
   (b) $f(x) = -2x - 10$
   (c) Undefined at $x = 1$.
   Not a function.
   (d) $f:x \rightarrow 4$

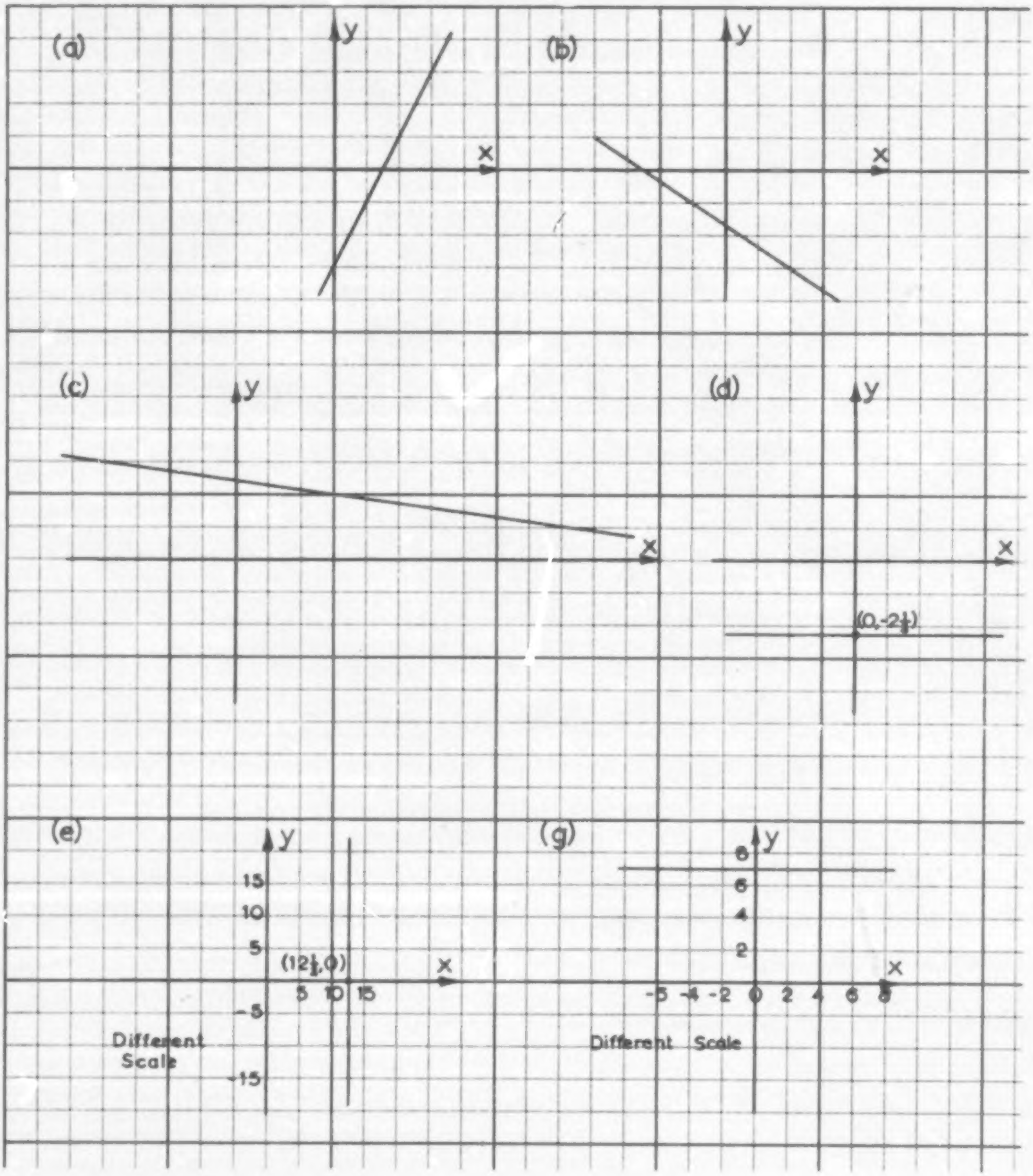
5. (a) $f:x \rightarrow -3x + 7$
   (b) $f:x \rightarrow -3x - 3$
   (c) $f(x) = -3x + 8$
   (d) $f(x) = -3x - 13$

6. (a) $f(3) = 5$
   (b) $f(3) = -3$
   (c) $f(3) = 4$

7. Yes. The slope of the line through P and Q is -2 and the slope of the line through P and S is -2. Two lines through the same point having the same slope coincide.

8. a,b,c,d,g define functions. f defines a function if $b \neq 0$. e fails since it has more than one value of the range for each value in the domain.

9.



10. (a) $f(-1) = 5$

$f(1) = -1$

(b) $f(-1) = -1$

$f(1) = -\frac{7}{3}$ or $-2\frac{1}{3}$

(c) $f(-1) = \frac{18}{7}$ or $2\frac{4}{7}$

$f(1) = \frac{16}{7}$ or $2\frac{2}{7}$

(d) $f(-1) = -\frac{7}{3}$ or $-2\frac{1}{3}$

$f(1) = -\frac{7}{3}$ or $-2\frac{1}{3}$

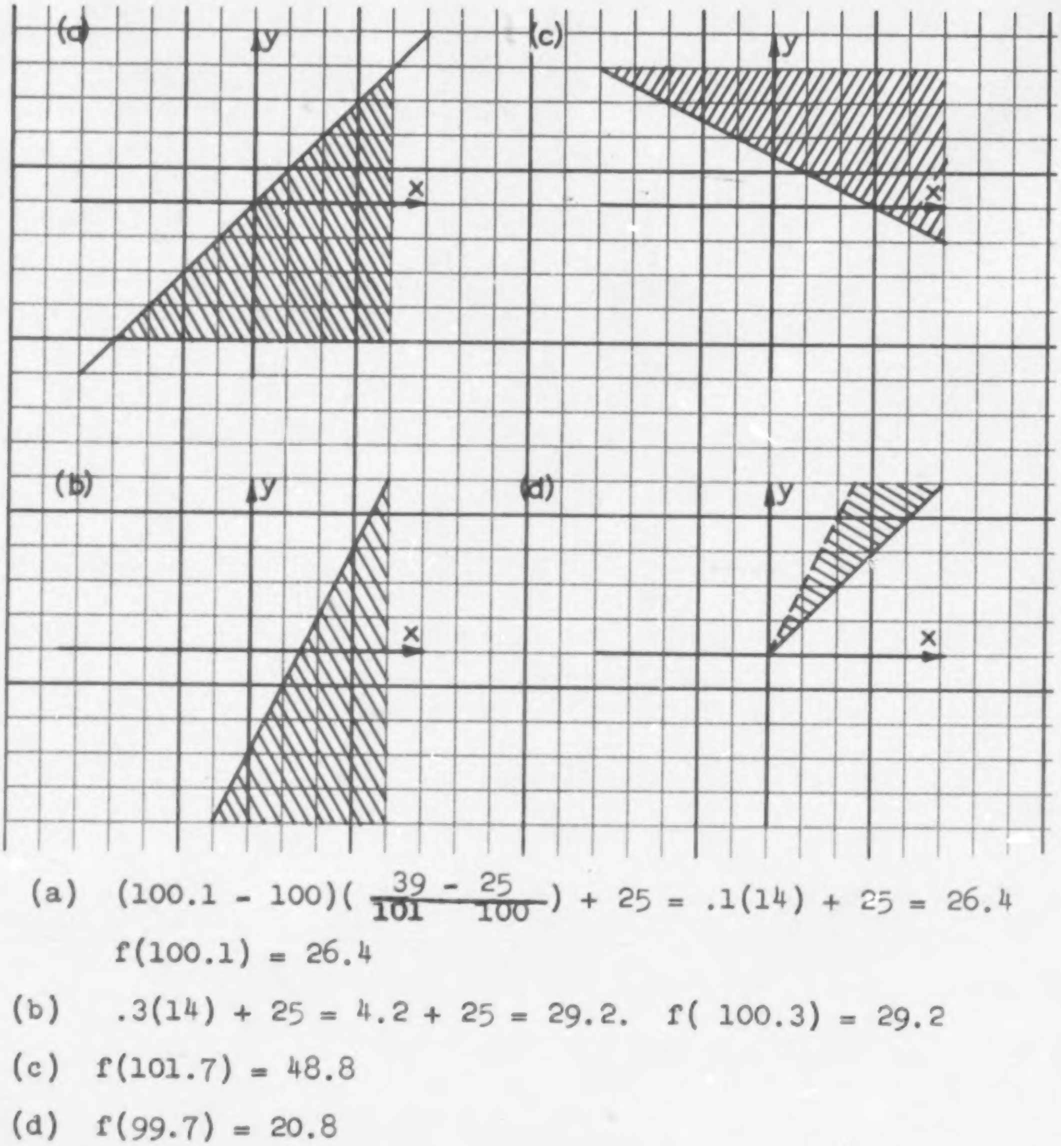
(e) f(x) is not defined (f) $f(-1) = \frac{a - c}{b}$

$f(1) = - \frac{a+c}{b}$

(g) $f(-1) = 7$

$f(1) = 7$

11.



12. (a) $(100.1 - 100)\left( \frac{39 - 25}{101 \quad 100} \right) + 25 = .1(14) + 25 = 26.4$

$f(100.1) = 26.4$

(b) $.3(14) + 25 = 4.2 + 25 = 29.2.$ $f(100.3) = 29.2$

(c) $f(101.7) = 48.8$

(d) $f(99.7) = 20.8$

13. (a) $f(53.3) = -44(.3) + 25 = 11.8$

(b) $f(53.8) = -10.2$

(c) $f(54.4) = -36.6$

(d) $f(52.6) = 42.6$

14.
$$\begin{cases} 2x + 7y + 1 = 0 \\ x - 2y + 8 = 0 \end{cases}$$
$$\begin{cases} 2x + 7y + 1 = 0 \\ 2x - 4y + 16 = 0 \end{cases}$$
$$11y = 15$$
$$\begin{cases} y = \frac{15}{11} \\ x = -\frac{58}{11} \end{cases}$$

$x - 3y + 4 = 0$

Slope $= \frac{1}{3}$

$y = \frac{1}{3}x + b$

$\frac{15}{11} = \frac{1}{3}(-\frac{58}{11}) + b$

$\frac{103}{33} = b$

$y = \frac{x}{3} + \frac{103}{33}$ or

$11x - 33y + 103 = 0$

15. The slopes of the lines AB and CD are $\frac{1}{4}$ and the slopes of the lines AD and BC are $-\frac{3}{2}$. Since the opposite sides are parallel (have the same slope), ABCD is a parallelogram.

16. (a) $C(4,8)$ (b) $C(5, -11)$

17. $f: x \longrightarrow 2x - 1$

$f(t+1) = 2(t+1) - 1 = 2t + 1$

$\therefore$ $P(t+1, 2t+1)$ is on the graph of $f$.

18. $f(0) = f(t-1)$ when $t = 1$. Then $f(0) = 4$

$f(8) = f(t-1)$ when $t = 9$. Then $f(8) = 82$

19. $f(0) = f(t-1)$ when $t = 1$. Then $f(0) = 2$

$f(8) = 82$

20. $f(x_1) = ax_1 + b$ , $f(x_2) = ax_2 + b$

$$f(x_1) - f(x_2) = ax_1 + b - (ax_2 + b) =$$

$$= ax_1 - ax_2 =$$

$$= a(x_1 - x_2)$$

Since $a < 0$ and $x_1 < x_2$ or $x_1 - x_2 < 0$,

$$a(x_1 - x_2) > 0$$

$\therefore$ $f(x_1) - f(x_2) > 0$ or $f(x_1) > f(x_2)$

## 2-2 Quadratic Functions

As stated in the introduction, this section contains the idea of linear approximation in its simplest possible form. (Only a single term is omitted to obtain the linear approximation.) Indeed, the most important ideas are explained in the first third of the section which should be studied with great care.

Some teachers may wish to stress Definition 2-2 given in the footnote. Our feeling is that normally it should be omitted on the first reading. In general, we believe that the treatment of limits should be kept on an intuitive level and we have resisted the impulse to formalize it except in a footnote.

The important concept of linear approximation should probably be illustrated by further numerical work. For example the meaning of

$$2x^2 + 3x + 4 \approx 3x + 4$$

might be made clear by showing that the omitted term $2x^2$ can be made as small a fraction of $3x$ as one wishes by taking $|x|$ small enough.

Thus

$$\frac{2x^2}{3x} = \frac{2}{3}x$$

can be made less then $0.001$ in absolute value by choosing $|x| < 0.0015$.

It is important that the student understand thoroughly the absolute value function, defined in Chapter 1. The graph in Figure TC 2a will help fix in mind the definition

$$|x| = x \quad \text{for} \quad x \geq 0$$
$$= -x \quad \text{for} \quad x < 0.$$

which is equivalent to the definition $|x| = \sqrt{x^2}$ which we shall prefer.

In the present section, the only property of the absolute value function used is that

$$|ab| = |a| \; |b|$$

This result is an almost immediate consequence of the second definition. Indeed,

$$\left| ab \right| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2} \; \sqrt{b^2}$$
$$= \left| a \right| \; \left| b \right| .$$

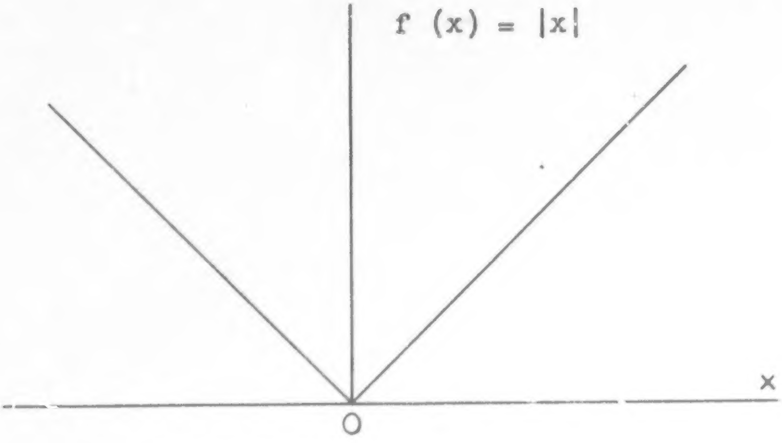
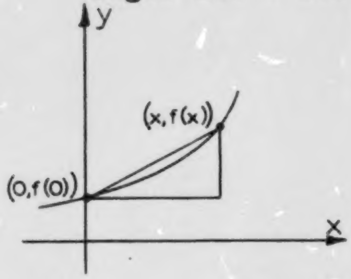


Fig. TC 2a

## Answers to Exercises 2 - 2a



1. $s(x) = \dfrac{f(x) - c}{x} = ax + b$

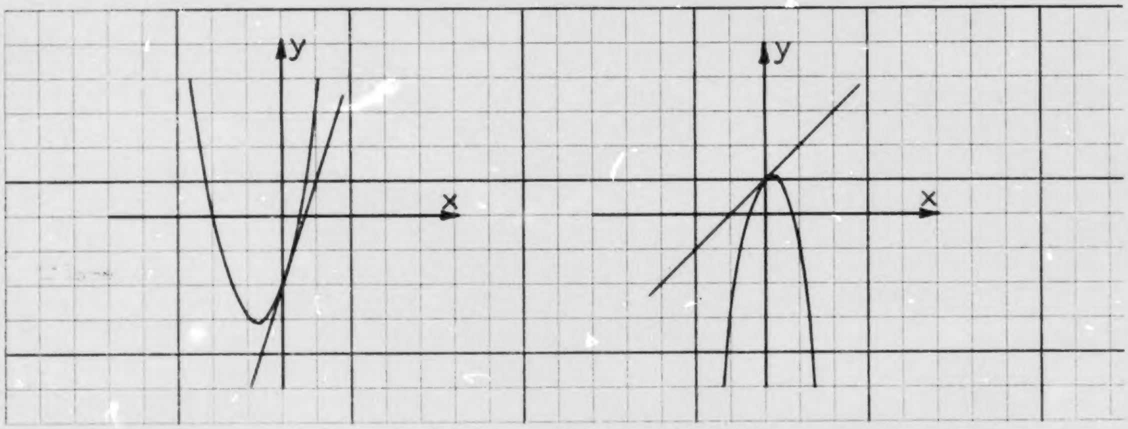
   $f(x) = ax^2 + bx + c$

   $f(x) - c = ax^2 + bx = x(ax + b)$

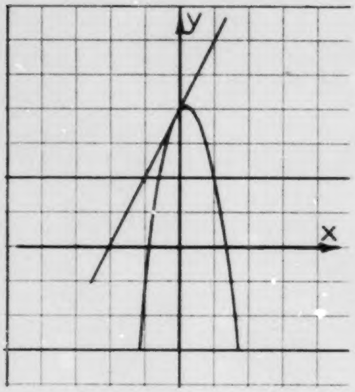
   $\dfrac{f(x) - c}{x} = ax + b$

   Slope of Secant $= \dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x) - c}{x} = s(x)$

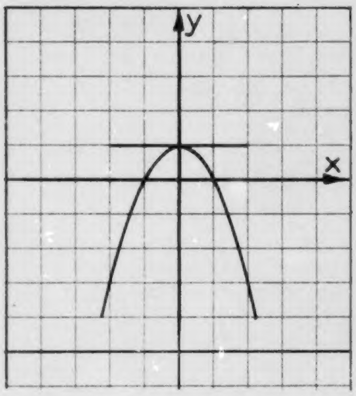
2. (a) $y = 3x - 2$ (b) $y = x + 1$



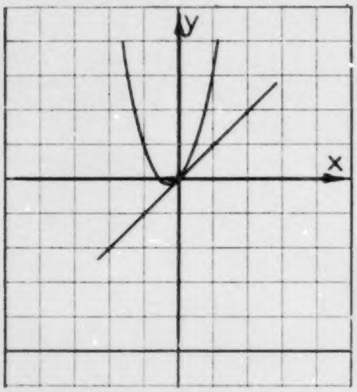
(c) $y = 2x + 4$



(e) $y = 1$



(d) $y = x$



3. $f(x) \approx bx + c$ near $x = 0$, Error = $E = ax^2$

(a) $f(x) \approx 3x - 2$ ; Error = $2x^2 = 0.0018$

(b) $f(x) \approx 1 + x$ ; Error = $-3x^2 = -0.0027$

(c) $f(x) \approx 2x + 4$ ; Error = $-4x^2 = -0.0036$

(d) $f(x) \approx x$ ; Error = $2x^2 = 0.0018$

(e) $f(x) \approx 1$ ; Error = $-x^2 = -0.0009$

4. (a) $s(x) = 2x + 3$ (d) $s(x) = 2x + 1$

(b) $s(x) = 1 - 3x$ (e) $s(x) = -x$

(c) $s(x) = - 4x + 2$

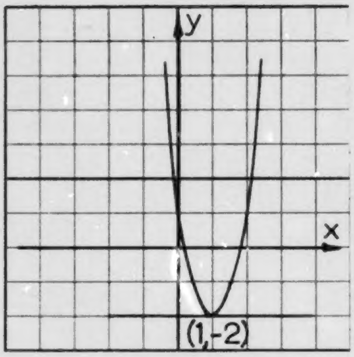
5. $f: x \longrightarrow 2x^2 + 5x + 6$; $y = 5x + 6$ ; Error $= 2x^2$

for $2x^2 < 0.01 \Rightarrow x^2 < 0.0050 \Rightarrow |x| < \sqrt{0.0050} \approx 0.07^+$

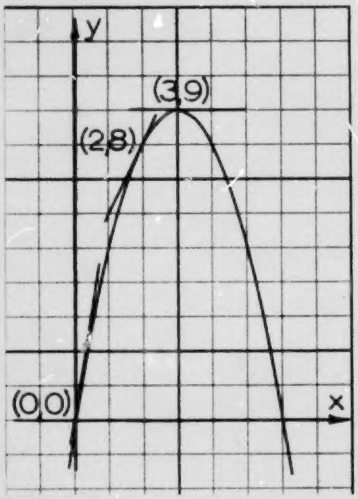
Answers to Exercises 2 - 2b Pages 105 - 106

1. (a) $y = -2$ at $(1, -2)$

(a)



(b)

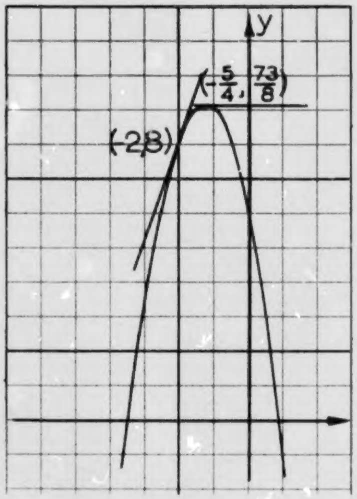


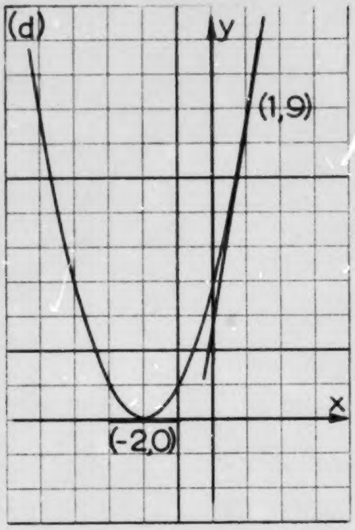
(b)

| Equation of tangent | At point |
|---|---|
| $y = 6x$ | $(0,0)$ |
| $y = 2x+4$ | $(2,8)$ |
| $y = 9$ | $(3,9)$ |

(c) $y = 3x + 14$ at $(-2,8)$

(c)





| (d) | Equation of tangent | At point |
|---|---|---|
| | $y = 6x+3$ | (1,9) |
| | $y = 0$ | (-2,0) |

2. (a) $y = (-6 + 6h)(x - h) + f(h)$

If $-6 + 6h = 0$; $h = 1$ and $f(1) = -2$

Range $f(x) \geq -2$

$\}$ See Exercise 1a for graph

(b) $y = (6 - 2h)(x - h) + f(h)$

If $6 - 2h = 0$; $h = 3$ and $f(3) = 9$

Range $f(x) \leq 9$

$\}$ See Exercise 1b for graph

(c) $y = (-5 - 4h)(x - h) + f(h)$

If $-5 - 4h = 0$; $h = -\frac{5}{4}$ and

$f(-\frac{5}{4}) = \frac{73}{8}$ or $9\frac{1}{8}$

$\}$ See Exercise 1c for graph

Range $f(x) \leq \frac{73}{8}$

(d) $y = (4 + 2h)(x - h) + f(h)$

If $4 + 2h = 0$; $h = -2$ and $f(-2) = 0$

Range $f(x) \geq 0$

$\}$ See Exercise 1d for graph

3. For $f: x \longrightarrow 2x^2 + 3x + 4$, we get

$$s(x) = \frac{f(x) - f(h)}{x - h} = \frac{2(x - h)^2 + (3 + 4h)(x - h) + \cancel{f(h)} - \cancel{f(h)}}{x - h} =$$

$$s(x) = 2(x - h) + (3 + 4h)$$

$$\lim_{x \to h} s(x) = \lim_{x \to h} [2(x - b) + (3 + 4h)] = 3 + 4h$$

For $|s(x) - (3 + 4h)| = |2(x - h)| = 2|x - h| < e$,

$|x - h| < \frac{e}{2}$.

## Answers to Exercises 2 - 2c

Page 109 - 110

1. (10) $f(x) = a(x - h)^2 + (2\,ah + b)(x - h) + f(h)$

$$s(x) = \frac{f(x) - f(h)}{x - h} = \frac{a(x - h)^2 + (2ah + b)(x - h) + \cancel{f(h)} - \cancel{f(h)}}{x - b} =$$

$$s(x) = a(x - h) + (2ah + b)$$

$$\lim_{x \to h} s(x) = a \cdot 0 + 2ah + b = 2ah + b$$

2. $s(x) = 2ah + b$

(a) $s(x) = 2 \cdot 1 \cdot h - 6 = 2h - 6$ ($= 0$ when $h = 3$, $f(3) = 2$)

$(3, 2)$ is a minimum since $a > 0$.

(b) $s(x) = 2(-5)h + 0 = -10h + 0$ ($= 0$ when $h = 0$, $f(0) = -2$)

$(0, -2)$ is a maximum since $a < 0$.

(c) $s(x) = 2(-1)h + 4 = -2h + 4$ ($= 0$ when $h = 2$, $f(2) = 0$)

$(2, 0)$ is a maximum since $a < 0$.

(d) $s(x) = 2(-4)h - 24 = -8h - 24$ ($= 0$ when $h = -3$, $f(-3) =$ $63$) $(-3, 63)$ is a maximum since $a < 0$.

(e) $s(x) = 2(4)h - 24 = 8h - 24$ ($=$ when $h = 3$, $f(3) = -9$)

$(3, -9)$ is a minimum since $a > 0$.

(f) $s(x) = 2(\frac{1}{8})h + 1 = \frac{1}{4}h + 1$ ($= 0$ when $h = -4$, $f(-4) = 0$)

$(-4, 0)$ is a minimum since $a > 0$.

3. $s(x) = 2ah + b$ (= 0 when $h = -\frac{b}{2a}$; $h - 1 = \frac{-b - 2a}{2a}$

$s(h-1) = \cancel{2a}\left(\frac{-b-2a}{\cancel{2a}}\right) + b = -2a$. If $a < 0$ then $f(-\frac{b}{2a})$ is a maximum. If $a > 0$ then $f(-\frac{b}{2a})$ is a minimum.

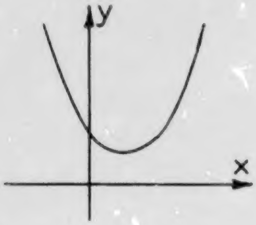
4. $-\frac{b}{2a} = 0$ when $b = 0$.

5. $b^2 - 4ac = 0$ is the condition for $f(x) = 0$ to have two equal roots. The maximun or minimum value of $f(x)$ will have to be zero, since the graph of $f$ will be tangent to the x-axis. In addition the quadratic formula gave the value of the two equal roots as $-\frac{b}{2a}$.

$$f\left(-\frac{b}{2a}\right) = \cancel{a}\left(\frac{b^2}{4a^{\cancel{2}}}\right) - \frac{b^2}{2a} + c = \frac{b^2 - 2b^2 + 4ac}{4a} = \frac{-(b^2 - 4ac)}{4a}\ .$$

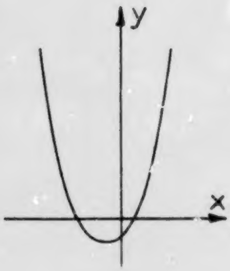
But since $b^2 - 4ac = 0$, $f(-\frac{b}{2a}) = 0$.

6. If $a > 0$ the graph of $f(x) = ax^2 + bx + c$ opens upward.



No real roots.

Minimum value is positive.



Two unequal real roots

Minimum value is negative

7. $f(1) = 1$ is the minimum point. Since this occurs when

$h = -\frac{b}{2a}$, $1 = -\frac{b}{2\cdot 3} \Longrightarrow b = -6$.

For $f(x) = 3x^2 - 6x + c$ to equal 1 when $x = 1$, $c = 4$.

## 2-3. Tangents to Polynomial Graphs

2-3 is an extension of the previous methods to polynomials of higher degree. An additional property of the absolute value function is required, namely that

$$|a + b| \leq |a| + |b|$$

Again using the definition,

$$|x| = \sqrt{x^2}$$

this is equivalent to

$$\sqrt{(a + b)^2} \leq \sqrt{a^2} + \sqrt{b^2} \qquad (1)$$

which is equivalent to

$$a^2 + 2ab + b^2 \leq a^2 + 2\sqrt{a^2}\sqrt{b^2} + b^2$$

and hence to

$$2ab \leq 2\sqrt{a^2}\sqrt{b^2}$$

or

$$ab \leq \sqrt{a^2}\sqrt{b^2} . \qquad (2)$$

Now equation (2) is easy to prove. If $a$ and $b$ have opposite signs $ab < 0$ and (2) holds with the $<$ sign. Otherwise, we have $ab = \sqrt{a^2}\sqrt{b^2}$ .

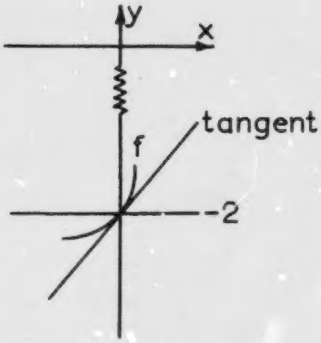
Hence in any case

$$ab \leq \sqrt{a^2}\sqrt{b^2} \qquad (2)$$

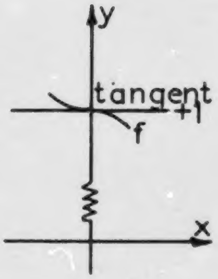
and therefore (1) holds.

Answers to Exercises 2 - 3. Page 116

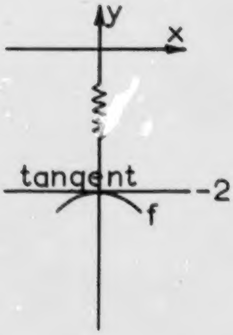
1. (a)



(c)

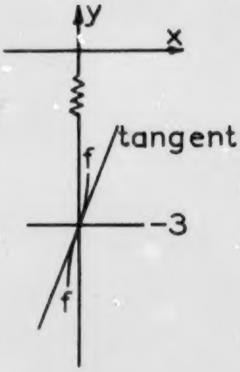


(b)

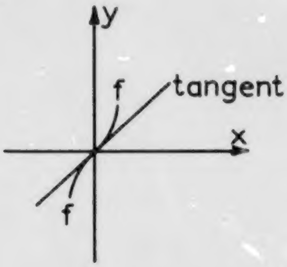


The sketches for this exercise are not in scale, as this is the best way to show the behavior of the function near the tangent.

(d)



(e)



*2. $s(x) = \frac{f(x) - d}{x} = ax^2 + bx + c$ (4)

$$E = |s(x) - c| = |ax^2 + bx|$$
$$\leq |ax^2| + |bx|$$
$$\leq |a| \cdot |x^2| + |b|\,|x|$$

Since $|x| < 1$, $|x^2| < |x|$, (if $x \neq 0$)

$$\therefore E < |a| \cdot |x| + |b| \cdot |x|$$
$$< (|a| + |b|) \cdot |x|$$

$\therefore$ the error $E$, can be made as small as we please by choosing $|x|$ small enough.

If we wish $E < e$, then we must take $|x| < \frac{e}{|a|+|b|}$

*3. $f(x) = ax^3 + bx^2 + cx + d$

$$f(x) - (cx + d) = ax^3 + bx^2$$
$$= x^2(ax + b)$$

As $|x|$ approaches $0$, $ax + b$ approaches $b$, since $|(ax + b) - b| = |ax| = |a| \cdot |x|$.

If we wish $|(ax + b) - b| < e$, we choose $|x| < \frac{e}{|a|}$

$\therefore ax^3 + bx^2$ has the same sign as $b$ for $|x|$ sufficiently small.

Answers to Exercises 2 - 4. Page 125

1. (a) Slope $= -5 - 2h + 12h^2$
   (b) Slope $= -15h^2 + 15h^4$
   (c) Slope $= -3 + 6h^5$
   (d) Slope $= h^2 + h^3$

2. $b_1 = a_1 + 2a_2h + 3a_3h^2$ and $b_2 = a_2 + 3a_3h$

   (a) Slope $= -12 + 6h + 6h^2$ ($= 0$ when $h = -2$, or $h = 1$)

   $(1, -14)$ is a relative minimum point. ($b_2 = 9$)
   $(-2, 13)$ is a relative maximum point. ($b_2 = -9$).

   (b) Slope $= -12h + 3h^2$ ($= 0$ when $h = 0$, or $h = 4$).

   $(0, 16)$ is a relative maximum point. ($b_2 = -6$).
   $(4, -16)$ is a relative minimum point. ($b_2 = +6$).

   (c) Slope $= -12 + 3h^2$ ($= 0$ when $h = +2$, or $h = -2$).

   $(2, 0)$ is a relative minimum point. ($b_2 = 6$).
   $(-2, 32)$ is a relative maximum point. ($b_2 = -6$).

   (d) Slope $= 12 + 6h - 6h^2$ ($= 0$ when $h = 2$, or $h = 1$).

   $(2, 27)$ is a relative maximum point. ($b_2 = -9$).
   $(-1, 0)$ is a relative minimum point. ($b_2 = +9$).

3. $b_1 = a_1 + 2a_2h + 3a_3h^2$

   $b_2 = a_2 + 3a_3h$

   $b_3 = a_3$

   (a) $f(x) = 2x^3 - 6x^2 + 6x - 1$. At $P(1,1)$

$$\left.\begin{array}{lll} b_1 & = 6 - 12h + 6h^2 & = 0 \\ b_2 & = -6 + 6h & = 0 \\ b_3 & = 2 & \neq 0 \end{array}\right\} \qquad \text{Point of Inflection}$$

   (b) $f(x) = 2x^3 - 6x + 5$. At $P(1,1)$

$$\left.\begin{array}{lll} b_1 & = -6 + 6h^2 & = 0 \\ b_2 & = 6h & = 6 \end{array}\right\} \qquad \text{Relative minimum}$$

(c) $f(x) = 2x^3 - 9x^2 + 12x - 4.$ At $P(1,1)$

$$\left.\begin{array}{l} b_1 = 12 - 18h + 6h^2 = 0 \\ b_2 = -9 + 6h = -3 \end{array}\right\} \quad \text{Relative maximum}$$

(d) $f(x) = 2x^3 - 3x^2 - 12x + 14$

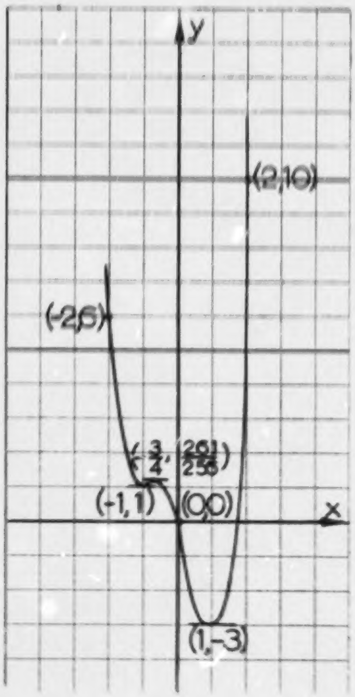
$b_1 = - 12 - 6h + 6h^2 = -12$ ] not one of the three.

4. $s(x) = - 3 - 4h + 3h^2 + 4h^3 = (-3 - 4h)(1 - h^2)$

$s(x) = 0$ when $h = + 1$, $h = -1$ and $h = - \frac{3}{4}$ .

Level points occur at $(-1,1)$, $(- \frac{3}{4} , \frac{261}{256})$ and $(1, -3)$.

The graph also goes through points $(-2, 6)$, $(0,0)$ and $(2,10)$.



## Answers to Exercises 2 - 5

Page 129

1. (a) $f' : x \longrightarrow 3 + 4x$

   (b) $g' : y \longrightarrow 2 - 6y + 3y^2$

   (c) $f' : x \longrightarrow -3 + 4x$

   (d) $h' : s \longrightarrow 4$.

   (e) $f' : t \longrightarrow -4t + 12t^2$

2. (a) $y = x$

   (b) $y = 0$

   (c) $y = 54 - 6r$

   (d) $y = 2s - 3$. (Since equation represents a straight line.)

   (e) Point is not on graph. Problem should read $t(t-3)(t+6)$.

3. (a) $(\frac{1}{2}, \frac{3}{4})$

   (b) $(\frac{1}{2}, \frac{5}{4})$

   (c) $(0,0)$, $(\frac{2}{3}, \frac{4}{27})$

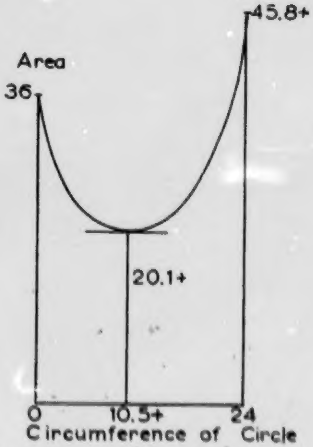
   (d) $(\frac{2\sqrt{3}}{3}, \frac{16\sqrt{3}}{9})$ $(-\frac{2\sqrt{3}}{3}, -\frac{16\sqrt{3}}{9})$

   (e) None, since $f'(x) = 0$ has no solution in real numbers.

### 2-6 Maximum and Minimum Problems

As previously stated it is desirable that some of these problems be taken up. Since the most difficult part of the problem may well consist in finding the function to maximize or minimize and since no general rule can be given for doing this, only a small number of exercises should be included in a lesson assignment.

It is possible for a function to take on its maximum or minimum value at an endpoint of the domain. In this case the slope function need not be zero. Exercise 4 is a case in point. The function has a graph which resembles the figure, with a maximum value at $x = 24$ where the slope is positive not zero. Note that the domain is restricted to the interval $[0, 24]$ by physical considerations. Note also that in Example 3, $V$ has a minimum of the same kind at $x = 0$.



The Appendix contains 3 supplementary sections: one on Mathematical Induction, and two on the determination of the area of a figure with one curved boundary. This material is suitable for a longer course. The sections on area are a natural extension of the present material. The section of Mathematical Induction can be taken in the better classes. Some of the starred problems could well be assigned to the very brightest students to keep them busy. The teacher ought not to give the solutions away until he is certain that the student has made a serious attack on the problem.

1. $V = (20 - 2x)^2 x = 4(x^3 - 20x^2 + 100x)$

   If $V' = 4(3x^2 - 40x + 100) = 0$, $x = 10$ or $\frac{10}{3}$. (10 is a minimum). For maximum, $V = \frac{10}{3} \cdot \frac{40}{3} \cdot \frac{40}{3} = \frac{16,000}{27} = 592.6^+$ cubic feet.

2. The number to be squared will be $\frac{2}{3}N$ and the other number is $\frac{N}{3}$.

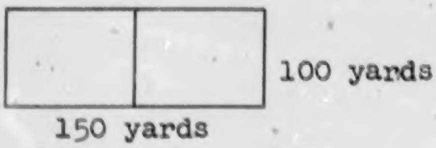
3. Fifty feet parallel with river; 25 feet on each side.

4. If $x$ is the length of the wire to be bent into the circle, the area is:

   $$A = \left(\frac{24 - x}{4}\right)^2 + \frac{x^2}{4\pi} = 36 - 3x + \frac{x^2}{16} + \frac{x^2}{4\pi}$$

   $A' = -3 + \frac{x}{8} + \frac{x}{2\pi}$ . If $A' = 0$, $x = \frac{24\pi}{\pi+4}$ .

   $x \approx 10.5^+$. This will give an area for both the square and circle of $20.1^+$ which is a minimum. To find the maximum, we must examine the range. (This is explained more fully in the notes.) It is when the entire length of wire is bent into the circle and is an area of $45.8^+$ .

5. 

   $A = x\left(\frac{600 - 3x}{2}\right)$

6. $G.I. = N[1.50 - (N - 10).03] = 1.80\ N - .03\ N^2$

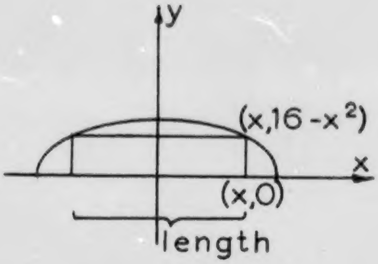
   where $N =$ number of thousands.

   If $(G.I.)' = 1.80 - .06\ N = 0.$ $N = 30.$

   30,000 labels will produce maximum gross income for printer.

7. The dimensions will be (height × width × length) = $(6 - 2\sqrt{3}) \times (4\sqrt{3}) \times (12 + 4\sqrt{3})$

8. $(\frac{8\sqrt{3}}{3}) \times (\frac{32}{3})$. The length of the rectangle is twice the distance of the x-value.



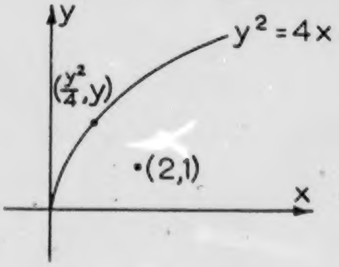
9. $D = \sqrt{(\frac{y^2}{4} - 2)^2 + (y - 1)^2}$

$K = D^2 = \frac{y^4}{16} - \cancel{y^2} + 4 + \cancel{y^2} - 2y + 1$

$K' = \frac{y^3}{4} - 2$ and if $K' = 0$,

$y^3 = 8 \Longrightarrow y = 2$.

So the point is (1, 2)



10. Seven weeks. The eight-week will not add to his profit. The number $7\frac{1}{2}$ is not an answer, (since $7\frac{1}{2}$ is not in our domain.)

11. Radius of inscribed right circular cylinder = $\frac{10\sqrt{6}}{3}$ inches.

Height of inscribed right circular cylinder = $\frac{20\sqrt{3}}{3}$ inches.

12. 35 trees per acre.

13. 28 inches.

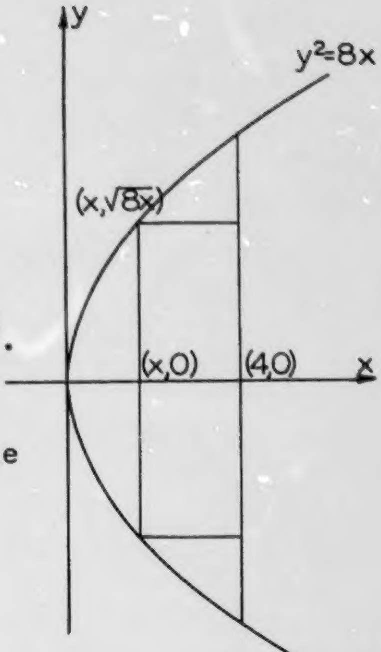
14. 2.5 inches deep.

15. July 15.

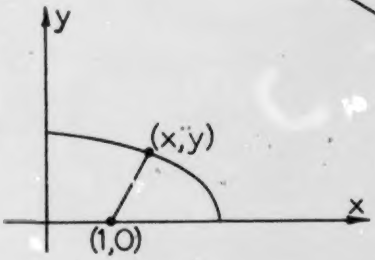
16. $A = (\frac{p - 2x}{2})x = \frac{p}{2}x - x^2$.

$A' = \frac{p}{2} - 2x$; $A' = 0$ when $x = \frac{p}{4}$.

If each side is $\frac{1}{4}p$ it is a square.



17. Area = $(4 - x)\sqrt{8x}$. This is not a polynomial so we set $K = A^2$.
$K = (4 - x)^2 \cdot 8x$. If we maximize K, we get the maximum of A.
$K = (16 - 8x + x^2)8x = 128x - 64x^2 + 8x^3$.
$K' = 24x^2 - 128x + 128$. If $K' = 0$, $x = 4$ or $\frac{4}{3}$. At $x = 4$, our rectangle having zero area would be a minimum.
If $x = \frac{4}{3}$ our rectangle would have a base $= \frac{4}{3}$ and a height $= \frac{8}{3}\sqrt{6}$.



18. Distance = $\sqrt{(x - 1)^2 + y^2}$

$$\sqrt{x^2 - 2x + 1 + y^2}$$

Since $y^2 = \dfrac{8 - x^2}{4}$

Distance = $\sqrt{x^2 - 2x + 1 + \frac{8-x^2}{4}} = \sqrt{\dfrac{3x^2 - 8x + 12}{2}}$

This is not a polynomial function, so we square it and obtain a polynomial function to maximize.

$K = \dfrac{3x^2 - 8x + 12}{4}$ and $K' = \dfrac{6x - 8}{4}$

For $K' = 0$, $x = \frac{4}{3}$ and $y = \frac{\sqrt{14}}{3}$

So point $(\frac{4}{3}, \frac{\sqrt{14}}{3})$ is nearest to (1, 0).

19. $\frac{4}{3} r$ .

20. Either \$112 or \$114 per month. (Again the answer obtained is not in our domain so that the nearest members of our domain have to be checked. See also Ex. 10.)

Answers to Miscellaneous Exercises of Chapter 2. Page 141 - 145.

1. $f: x \longrightarrow 2$ or $y = 2$.
2. When $a = b = 0$ and $c \in R$.
3. The point of intersection is $(\frac{1}{5}, -\frac{7}{5})$, so $f: x \longrightarrow -\frac{7}{5}$ .
4. The point of intersection is $(\frac{b-4}{a-5}, \frac{ab - 20}{a - 5})$, provided $a \neq 5$. If $a = 5$, they will be parallel, or if also $b = 4$, they will coincide.
5. $f_1: x \longrightarrow x + 3$ and $f_2: x \longrightarrow -x - 3$.
6. $10x + y - 7 = 0$ or $y = 7 - 10x$ and $a = -10$. This means for each unit increase in $x$, $y$ decreases by 10.
   If $x$ increases from 500 to 505, $y$ decreases 50.
   If $y$ decreases from -500 to -505, $x$ increases .5 or $\frac{1}{2}$ .
7. Slope of line $= \frac{3 - 1}{2 - (-1)} = \frac{2}{3}$ and line is $y = \frac{2}{3}x$.
8. Point of intersection is $(0, k)$, so line is $y = \frac{5}{6}x + k$.
9. $$\left.\begin{aligned} 6x + 3y - 7 = 0 &\Rightarrow y = \frac{7}{3} - 2x \\ y = -2x + 3 &\Rightarrow y = 3 - 2x \end{aligned}\right\}$$
   Lines are parallel and $\frac{2}{3}$ of a unit apart on the y-axis.
   Half of this distance is $\frac{1}{3}$ so the line is $y = \frac{8}{3} - 2x$.

10. $2y = x + 3 \Longrightarrow y = \frac{x}{2} + \frac{3}{2}$. Slope $= \frac{1}{2}$.

Slope of $\perp$ is $-2$. So line is $y = -2x + 18$.

11. $y = f_1(t) = t - 10$ when $t$ is in minutes.

$y = f_2(t) = 60(t - \frac{1}{6})$ when $t$ is in hours.

Domain of $t$ for $f_1$ is $\{t : t \geq 10$ and $t$ is a natural number.$\}$; for $f_2$ it is $\{t : t \geq \frac{n}{60}$ and $n \geq 10$ and $n$ is a natural number.$\}$

12. (a) AC: $y = \frac{7}{12}x$

(b) BD: $y = 14 - \frac{7}{4}x$

(c) Intersection is point $(6, 3\frac{1}{2})$

13. (a) AC: $y = \frac{y_2}{x_2}x$

(b) BD: $y = \frac{x_1 y_2}{2x_1 - x_2} - \frac{y^2}{2x_1 - x_2}x$ or

$\frac{y_2}{2x_1 - x_2}(x_1 - x)$

(c) Intersection is point $(\frac{x_2}{2}, \frac{y_2}{2})$

14. $f(x) = ax^2 + bx + c$

$f(0) = a(0) + b(0) + c = 0 \Longrightarrow c = 0$

$f(1) = a(1) + b(1) = 0 \Longrightarrow -a = b$

So $f(x) = ax^2 - ax$.

15. $f' : x \longrightarrow 2x + 5$. If $f' = -5 = 2x + 5$ $x = -5$

Point $(-5, 4)$

16. $f' : x = 2ax + b$ If $f'(0) = 4$ $b = 4$

$f : x \longrightarrow ax^2 + bx + c$ $f : 0 = -3 = c$

All quadratic functions passing through $P(0,-3)$ and having a slope $= 4$ at this point are given by

$f : x \longrightarrow ax^2 + 4x - 3$.

17. $S(PQ_1) = \frac{16 - 5}{1} = 11$ $\quad S(PQ_5) = \frac{14 - 5}{-1} = -9$

$S(PQ_2) = \frac{5.2 - 5}{.1} = 2$ $\quad S(PQ_6) = \frac{5 - 5}{-.1} = 0$

$S(PQ_3) = \frac{5.011 - 5}{.01} = 1.1$ $\quad S(PQ_7) = \frac{4.991 - 5}{-.01} = .9$

$S(PQ_4) = \frac{5.00101 - 5}{.001} = 1.01$ $\quad S(PQ_8) = \frac{4.99901 - 5}{-.001} = .99$

18. $f:x \rightarrow 2x^2 - 7x - 4$ and $f':x \rightarrow 4x - 7$

$x_2 = x_1 + 2 \quad f':x_2 = f':x_1 + 2 = 4x_1 + 8 - 7$

$f':x_1 =$

$f'(x_2) - f'(x_1) \quad = \quad \underline{4x_1 \quad - 7}$

$+8$

19. $f:x \rightarrow ac^2 + bx + 8 \quad ; \quad P(-3,2); \quad f':x \rightarrow 2ax + b$

$$\begin{cases} 9a - 3b + 8 = +2 \\ -6a + b = -1 \end{cases} \qquad \begin{cases} 9a - 3b = -6 \\ \underline{-18a + 3b = -3} \end{cases} \qquad \begin{cases} a = 1 \\ b = 5 \end{cases}$$

$-9a = -9$

20. $f:x \rightarrow ax^2 + bx + c \qquad f':x \rightarrow 2ax + b$

$$\begin{cases} f:6 \rightarrow 36a + 6b + c = 0 \\ f:2 \rightarrow \underline{4a + 2b + c = 0} \end{cases}$$

$f':6 \rightarrow 12a + b = 5$

$f':2 \rightarrow 4a + b = ?$

$$\begin{cases} 32a + 4b = 0 \\ 12a + b = 5 \end{cases}$$

$5 - 10 = -5$

$b = 5 - 12a$

$32a + 20 - 48a = 0$

$16a = 20$

$$\begin{cases} a = \frac{5}{4} \\ b = -10 \end{cases}$$

Answer: Slope at $(2,0)$ is $-5$

21. $y = (2ah + b)(x - h) + f(h)$ and $f: x \longrightarrow x^2 - 4x + 3$

If $h = 2$, $y = -1$

If $h = 4$, $y = 4(x - 4) + 3 = 4x - 13$

Error $f(x) - y = (x - h)^2$

$(2.01 - 2)^2 = .0001$

$(4.01 - 4)^2 = .0001$

If $(x - h)^2 < 0.01$

then $|x - h| < 0.1$

22. $f: x \longrightarrow 5x^2 + (-10h + 1) + 5h^2 - h + 3$

23. Given: $f(x) = a(x - h)^2 + b(x - h) + c$ and

$f(x) = A\,x^2 + bx + f(k)$

By algebraic manipulation we get respectively:

$f(x) = ax^2 + (b - 2ah)x + ah^2 - bh + c$ and

$f(x) = A(x - h)^2 + (2Ah + B)(x - h) + f(h)$

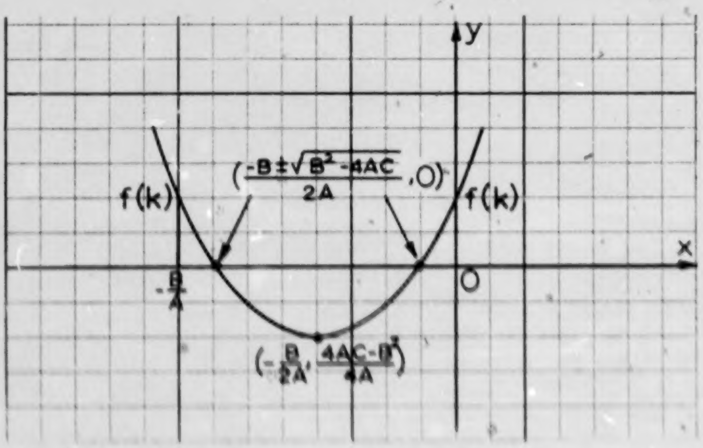
And so $A = a$

$B = b - 2ah$

$b = 2Ah + b$

$f(k) = Ak^2 + Bk + f(k)$ and

$0 = k(Ak + B) \Longrightarrow k = 0$ or $k = -\frac{B}{A} = \frac{2ah - b}{a}$



The primary purpose of this exercise was to have the student recognize without manipulation that $k = 0$. This could be obtained by a bright student from the second equation. It is interesting to see that two values of $k$ can be obtained. This need not be completely explained to the students.

24. If $f{:}x \longrightarrow 2x^3 - 5x^2 + x + 3$, write $f(x)$ in powers of $x + 2$

What must be done to $x$ to get it = $(x + 2)$ and something?

Use $x = (x + 2) - 2$.

$$f{:}(x + 2) - 2 \longrightarrow 2[(x + 2) - 2]^3 - 5[(x+2) - 2]^2 + [(x+2) - 2] + 3 =$$

$$f{:}x \longrightarrow 2(x + 2)^3 - 17(x+2)^2 + 45(x + 2) - 35$$

25. Write $f{:}x \longrightarrow a(x - h)^2 + b(x - h) + c$ in powers of $x - 1$

$$f{:}(x - 1) + 1 \longrightarrow a[(x - 1) + 1 - h]^2 + b[(x - 1) + 1 - h] + c =$$

$$f{:}x \longrightarrow a(x - 1)^2 + (2a - 2h + b)(x - 1) + \underline{a(1 - h)^2 \quad b(1 - h) + c}$$

$$f{:}x \longrightarrow a(x - 1)^2 + (2a - 2h + b)(x - 1) + \qquad f(1)$$

26. $f{:}x \longrightarrow ax^2 + bx + c$ has zeros at $x = 1_1$ and $x = x_2$

So, $ax_1^2 + bx_1 + c = 0$

and $\underline{ax_2^2 + bx_2 + c = 0}$

$$a(x_1^2 - x_2^2) + b(x_1 - x_2) = 0 \Longrightarrow a(x_1 + x_2) = -b \text{ if } x_1 \neq x_2$$

But $f'{:}x \longrightarrow 2ax + b = 0$ for $x = \dfrac{-b}{2a} = \dfrac{a(x_1 + x_2)}{2a}$ .

$\therefore$ $f$ has a minimum at $x = \dfrac{x_1 + x_2}{2}$

27. If $f:x \to a_2x^2 + a_1x + a_o = b_2(x - h)^2 + b_1(x - h) + b_o$ where $b_2 = a_2$, $b_1 = a_1 + 2a_2h$, $b_o = a_2h^2 + a_1h + a_o$. The necessary condition for a point of inflection is that $b_2 = 0$. But this means $a_2 = 0$ which means that $f$ is not a quadratic function.

28. $f:x \to ax^2 + bx + c$ ; $f:0 \to 0 = c$

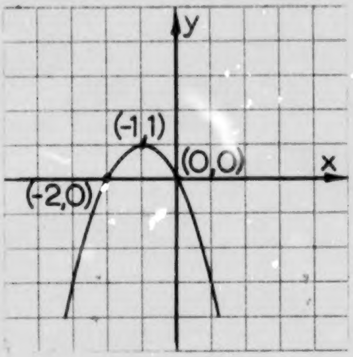
$f':x \to 2ax + b$ $\quad f':(-1) \to -2a + b = 0$

or $\underline{\quad b = 2a \quad}$

$f:(-1) = a - b = 1$

$a = -1$

(a) $f:x \to -x^2 - 2x$



(b) $g:x \to x^2 + 2x + 2$ is symmetric with $f$ about the line $y = 1$.

29. (a) $A = x(120 - 3x)$

$A' = 120 - 6x$. If $A' = 0$, $x = 20$; $A = 20(60) = 1200$ square feet.

(b) $A = x(\frac{120 - 2x}{2}) = 60x - x^2$

$A' = 60 - 2x$. If $A' = 0$, $x = 30$; $A = 30(30) = 900$ square feet.

30. $d = 88t - 16t^2$. If $d = 0$, $t = 0$ or $\frac{11}{2}$ seconds.

$d' = 88 - 32t$. If $d' = 0$, $t = \frac{11}{4}$ seconds.

Answer, in $\frac{11}{4}$ seconds, it will reach a height of 121 feet. It will hit the ground in $\frac{11}{2}$ seconds.

31. This is not a polynomial and at $x = 1$ blows up. It therefore should not have been included in this set of exercises. Since at $x = 0$, it is approximated by the polynomial function $x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$. The equation of the tangent is thus $y = x + 1$.

32. $y = x^2 - 3x + 4$. At $x = 0$, the slopes will be the same.

33. $f: x \longrightarrow ax^3 + bx^2 + cx + d$

$$\begin{array}{lllll} f: 1 \longrightarrow a & + b & + c & + d & = -2 \\ \underline{f: -1 \longrightarrow a} & \underline{+ b} & \underline{- c} & \underline{+ d} & \underline{= 2} \\ & 2b & & + 2d & = 0 \Longrightarrow b = -d \\ 2a & & + 2c & & = -4 \Longrightarrow a = -c - 2 \text{ or } c = -a - 2 \end{array}$$

but $f: 0 = 0$. So $d = 0$ and $b = 0$.

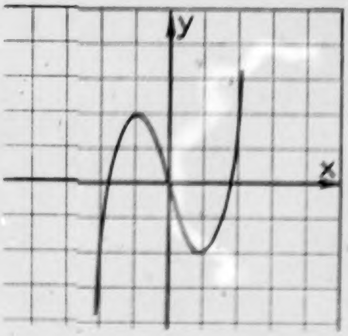
$f: x \longrightarrow ax^3 - x(a + 2)$

$f': x \longrightarrow 3ax^2 - (a + 2)$ and $f': 1 \longrightarrow 3a - (a + 2) = 0$

$2a = 2$ ; $a = +1$ ; and $c = -3$.

So: $f: x \longrightarrow x^3 - 3x$

$f': x \longrightarrow 3x^2 - 3$



$$\left.\begin{array}{l} f': h \longrightarrow 3h^2 - 3 \\ f': h \longrightarrow 3h^2 - 3 \end{array}\right\} \therefore$$ slopes at $x = h$ and $x = -h$ are the same, which means the graph is symmetric with respect to the origin.

If $f: x \longrightarrow x^3 - 3x = (x - h)^3 + 3h(x - h)^2 + (3h^2-3)(x-h)+f(h)$.

As the tangent is $y = (3h^3 - 3)(x - h) + f(h)$, $f(x)$ is going to cross this tangent line since error will depend on the sign of the coefficient of $(x - h)^2$. $3h$ is negative for $h < 0$ and then positive for $h > 0$. So $(0,0)$ is a point of inflection.

34. $g(x) = (x - 3)^3 - 3(x - 3)$

$g'(x) = 3(x - 3)^2 - 3$; $g'(3) = -3$

$$\left.\begin{aligned} g'(3 + k) &= 3(3 + k - 3)^2 - 3 = 3k^2 - 3 \\ g'(3 - k) &= 3(3 - k - 3)^2 - 3 = 3k^2 - 3 \end{aligned}\right\}$$

Since $g(x) - y = (x - 3)^3$

For $x < 3$ this difference is negative, and $x > 3$ has its difference positive. This means the graph of $g$ crosses the tangent at $x = 3$. This is a point of inflection.

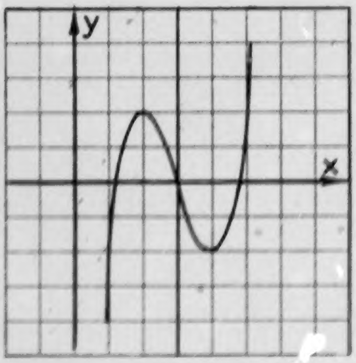


Figure to exercise 34

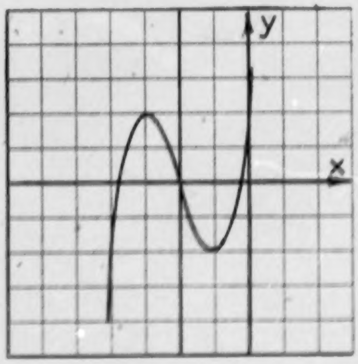


Figure to exercise 35

35. $$\left.\begin{array}{l} f:x \longrightarrow x^3 - 3x \\ g:x \longrightarrow (x - 3)^3 - 3(x - 3) \end{array}\right\} \qquad g(x) = f(x - 3)$$

$$g_1:x \longrightarrow (x + 2)^3 - 3(x + 2) = x^3 + 6x^2 + 9x + 2$$

36. $f$ is symmetric with respect to $(0,0)$, the point of inflection.

$g$ is symmetric with respect to $(3,0)$, the point of inflection.

*37. $f:x \longrightarrow a_3x^3 + a_2x^2 + a_1x + a_o$

$f':x \longrightarrow 3a_3x^2 + 2a^2x + a_1$

If $f':x_1 \longrightarrow 3a_3x_1{}^2 + 2a_2x_1 + a_1 = 0$

$f':x_2 \longrightarrow 3a_3x_2{}^2 + 2a_2x_2 + a_1 = 0$

$$3a_3(x_1{}^2 - x_2{}^2) + 2a(x_1 - x_2) = 0$$

If $x_1 \neq x_2$, $\quad a_2 = \dfrac{-3a_3(x_1+x_2)}{2}$

For a point of inflection the coefficient $b_2$ of $(x - h)^2$ term must = 0.

If $b_2 = a_2 + 3a_3h$ it follows that

when $h = x$ and $b_2 = 0$

$a_2 = - 3a_3x$ and substituting in $a_2$ above we get for $x$:

$$-3a_3x = \frac{-3a_3(x_1 + x_2)}{2}$$

$$x = \frac{x_1 + x_2}{2}$$

## Illustrative Test Questions for Chapter 2

1. Find a linear function $x \longrightarrow f(x)$ such that $f(2) = 3$ and $f(3) = 2f(4)$.

2. If a linear function has a slope of $\frac{3}{2}$ and the value $-3$ when $x = 2$, what is the value of the function when $x = 7$?

3. If $f(x)$ is a linear function such that $f(5) - f(2) = 4$, what is the slope of $f(x)$?

*4. Find the linear function whose graph passes through the points whose coordinates are of the form

$$\big((t + 3)(t - 2),\ (t + 4)(t - 3)\big).$$

5. Find the value of k for which the point $(k, 2k)$ lies on the line through the points $(3,-2)$ and $(5,4)$.

6. For each function below, write the equation of the tangent to the graph at $(0, f(0))$ and sketch the graphs of the function and the tangent in the vicinity of this point:

   (a) $f : x \longrightarrow f(x) = 3x^2 - 2x + 1$

   (b) $f : x \longrightarrow f(x) = 4 - x^2$

7. For what range of values of x is the error less than 0.01 when $4x^2 + 3x - 2$ is replaced by its tangent linear approximation at $x = 0$?

8. Given the function $x \longrightarrow f(x) = 4 - 3x + 2x^2$, find the equation of the tangent at $(h, f(h))$ and find the value of h for which the tangent is horizontal. Sketch the graph of the function and its horizontal tangent.

9. Find the value of k for which the maximum point of the graph of $f(x) = 2k + 3x - 5x^2$ has the same x and y coordinates.

10. Classify each of the points $(1,0)$, $(2,-2)$ and $(3,-4)$ on the

graph of $f(x) = x^3 - 6x^2 + 9x - 4$ as either a relative maximum, a relative minimum, or a point of inflection.

11. Find the points on the graph of $f(x) = 2x^3 - 9x^2 + 18x - 10$ where the slope is 1.

*12. Find an equation of the tangent to the graph of $f(x) = x^3 + 3x^2 - 4x - 3$ at its point of inflection.

**13. If a 36 inch piece of wire is cut into 2 pieces to be formed into a square and an equilateral triangle, what is the maximum total area of these 2 figures?

14. Express the polynomial $2x^3 - 5x^2 + 3x - 4$ in the form $a_0 + a_1(x - 3) + a_2(x - 3)^2 + a_3(x - 3)^3$.

15. Find the quadratic polynomial whose graph passes through the origin and which has a relative maximum at (2, 3).

16. A right triangle whose hypotenuse is 5 is rotated about one leg to form a right circular cone. What is the largest volume which the cone can have?

Answers to Illustrative Test Questions

1. $x \longrightarrow f(x) = ax + b$

$f(2) = 2a + b = 3$

$f(3) = 3a + b = 2f(4) = 2(4a + b) = 8a + 2b$

If $3a + b = 8a + 2b, \qquad -5a = b$

$2a - 5a = 3 \Longrightarrow -3a = 3 \Longrightarrow a = -1$

and $b = 5$.

$f(x) = 5 - x$

2. $f(x) = \frac{3}{2}x + b$

$f(2) = -3 = \frac{3}{\cancel{2}}(\cancel{2}) + b$

$b = -6$

$f(7) = \frac{3}{2}(7) - 6 = \frac{21}{2} - 6 = \frac{9}{2}$

3. $f(x) = ax + b$

$f(5) - f(2) = 5a + b - (2a + b) = 3a = 4$

Slope $= a = \frac{4}{3}$

*4. $(t^2 + t - 6,\ t^2 + t - 12) = (x, y)$

$(x\ ,\ x - 6) = (x, y)$

and $y = x - 6$.

5. $\frac{4 - (-2)}{5 - 3} = \frac{2k - (-2)}{k - 3}$

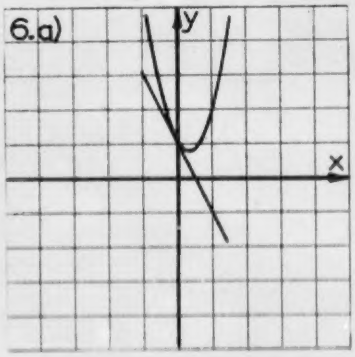
$3 = \frac{2k + 2}{k - 3}$

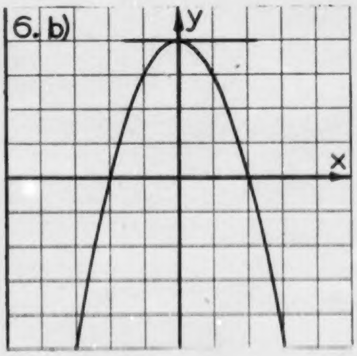
$3k - 9 = 2k + 2$

$k = 11$

6. a) $f:x \longrightarrow f(x) = 3x^2 - 2x + 1$

at (0, 1) tangent is $y = -2x + 1$





b) $f:x \longrightarrow f(x) = 4 - x^2$

at (0, 4) tangent is $y = 4$.

7. $f(x) = 4x^2 + 3x - 2$

$y = 3x - 2$ is tangent at point $(0, -2)$

Error $= 4x^2$ If Error is to be less than 0.01

$4x^2 < 0.01$

$x^2 < 0.0025$ Answer, $-0.05 < x < 0.05$

$|x| < 0.05$

8. $f(x) = 4 - 3x + 2x^2$

Let $x = (x - h) + h$

and $x^2 = (x - h)^2 + 2h(x - h) + h^2$

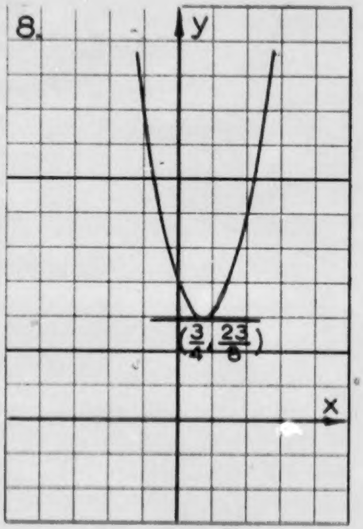
Then $f(x) = 4 - 3(x - h) - 3h + 2(x - h)^2 + 4h(x - h) + 2h^2$

$f(x) = 4 - 3h + 2h^2 + (4h - 3)(x - h) + 2(x - h)^2$

Tangent at $\big(h, f(h)\big)$ will be $y = (4h - 3)(x - h) + f(h)$

For the tangent to be horizontal

$4h - 3 = 0$ and $h = \frac{3}{4}$



9. $f(x) = 2k + 3x - 5x^2$

$f'(x) = 3 - 10x = 0 \Longrightarrow x = .3$

$.3 = 2k + 3(.3) - 5(.3)^2$

$.3 = 2k + .9 - .45$

$-.15 = 2k$

$k = -.075$

10. $f(x) = x^3 - 6x^2 + 9x - 4$

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0$ when $x^2 - 4x + 3 = 0$ and $x = 1$ or $x = 3$

$f''(x) = 6x - 12$ at 1 is negative

at 3 is positive

at 2 is zero.

So: (1, 0) is a relative maximum.

(2, -2) is a point of inflection.

and (3, -4) is a relative minimum.

11. $f(x) = 2x^3 - 9x^2 + 18x - 10$

$f'(x) = 6x^2 - 18x + 18$

When does $f'(x) = 1$? ; $6x^2 - 18x + 18 = 1$

For $6x^2 - 18x + 17 = 0$

x would have to be imaginary. So there are no points at which the slope of the tangent is 1.

12. $f(x) = x^3 + 3x - 4x + 3$

$f'(x) = 3x^2 + 6x - 4$

For $f'(x) = 0$, $x = \frac{-6 \pm \sqrt{36 + 48}}{6} = \frac{-6 \pm \sqrt{84}}{6} = -1 \pm \sqrt{84}$

Point of inflection $= \frac{x_1 + x_2}{2}$ where $x_1$ and $x_2$ are maximum and minimum points.

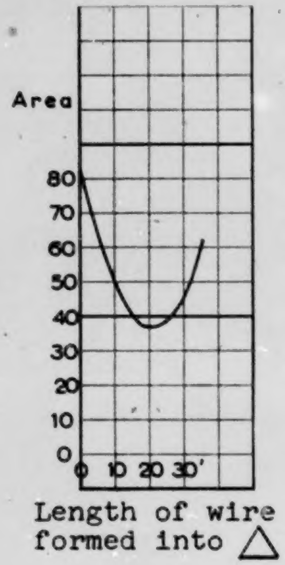
$$\frac{(-1 + \sqrt{84}) + (-1 - \sqrt{84})}{2} = -1$$

$(-1, f(-1)) = (-1, 9)$ Slope at $(-1, 9)$ is $f'(-1) = -7$

$y = ax + b$ through $(-1, 9)$ with slope of $-7$ is

$y = -7x + 2$

**13.



This example is like the one in 2-6, Exercise 4. This is discussed in the Commentary. If $x$ represents the piece of wire to be cut and formed into a triangle, the area will be

$$A = \frac{36 - x}{4}^2 + \left(\frac{x}{3}\right)^2 \frac{\sqrt{3}}{4}$$

This gives a relative minimum at $A' = 0$.

14. Let $x = (x - 3) + 3$. Then

$$2x^3 - 5x^2 + 3x - 4 = 2(x - 3)^3 + 13(x - 3)^2 + 27(x - 3) + 22$$
$$= 22 + 27(x - 3) + 13(x - 3)^2 + 2(x - 3)^3.$$

15. $f(x) = ax^2 + bx + c$

$f(0) = \underline{c = 0}$

$f(x) = ax^2 + bx$

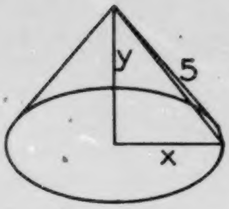
$f'(x) = 2ax + b = 0$ and $x = -\frac{b}{2a} = 2$

$-b = 4a$ or $b = -4a$

$f(2) = 4a - 8a = 3$ and $a = -\frac{3}{4}$ so $b = 3$

$f(x) = -\frac{3}{4}x^2 + 3x$

16.



$x^2 + y^2 = 5 \implies x^2 = 5 - y^2$

Volume $= \pi x^2 \cdot \frac{y}{3} = (5 - y)^2 \ \frac{\pi y}{3} =$
$\frac{5\pi y}{3} - \frac{\pi y^3}{3}$

$V' = \frac{5\pi}{3} - \pi y^2$ If $V' = 0$, $y = \sqrt{\frac{5}{3}}$ and $x^2 = \frac{10}{3}$ . .So,
Volume $= \frac{10\pi}{9}\sqrt{\frac{5}{3}}$ or $\frac{10\pi\sqrt{15}}{27}$

# Chapter 3

## Introduction

This chapter covers material on the solution of polynomial equations, usually included in a course in advanced algebra. It differs from the conventional treatments in three respects. (1) It uses somewhat more precise terminology and a greater amount of symbolism than is customary. (2) On the other hand, the treatment of synthetic division is more intuitive than usual. (3) For the computation of irrational roots, Newton's method has been used rather than Horner's method. This matter is discussed more fully in the comments on Section 3-8.

Some historical information has been included in the text with detailed references to generally accessible books. This subject offers unusual opportunity to give the student an interesting and understandable introduction to the history of mathematics.

The appendix contains a section on the importance of polynomials which includes a treatment of the Lagrange interpolation formula. This work is suitable for a longer course or for superior students.

## 3-1. Introduction and Notation

This section gives an overview of the history of the problem of determining the zeros of a given polynomial. It also includes an extended treatment of the use of summation notation.

The summation symbol is used to represent a polynomial of degree n in abbreviated form. (See page 149.) In this connection it should be noted that, strictly speaking, $x^0$ is not defined at $x = 0$. Since, however, $x^0 = 1$ for all $x \neq 0$, we may make the graph of $x^0$ continuous by supplying the point (0, 1). In this way by taking $x^0 = 1$ for all x in the summation notation for $P_n(x)$, we include the constant term.

The zero polynomial is said to have no degree. (See page 153.) This is consistent with Theorem 3-8 on page 157. The problem involved in assigning a degree to the zero polynomial is suggested by Exercise #6 in Miscellaneous Exercises page 217.

Answers to Exercises 3-1 Page 154

1. $\sum_{i=1}^{7}(2i - 1) = 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$ or

$$= \sum_{i=1}^{7} 2i - \sum_{i=1}^{7} 1$$

$$= (2 + 4 + 6 + 8 + 10 + 12 + 14) - 7 = 49$$

2.

a) $$\sum_{i=1}^{5} (x_i - m)^2 = \sum_{i=1}^{5} (x_i^2 - 2x_i m + m^2)$$

$$= \sum_{i=1}^{5} x_i^2 - \sum_{i=1}^{5} 2x_i m + \sum_{i=1}^{5} m^2$$

$$= \sum_{i=1}^{5} x_i^2 - 2m \sum_{i=1}^{5} x_i + 5m^2$$

$$= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$

$$- 2m(x_1 + x_2 + x_3 + x_4 + x_5) + 5m^2 .$$

b) if $m = \frac{1}{n} \sum_{i=1}^{n} x_i$ then $mn = \sum_{i=1}^{n} x_i$

$$\sum_{i=1}^{n} (x_i - m)^2 = \sum_{i=1}^{n} (x_i)^2 - 2m \sum_{i=1}^{n} x_i + nm^2$$

$$= \sum_{i=1}^{n} (x_i)^2 - 2m^2 n + m^2 n$$

$$= \sum_{i=1}^{n} (x_i)^2 - m^2 n$$

$$= \sum_{i=1}^{n} (x_i^2 - m^2)$$

3. $\sum_{i=1}^{5} ka_i = ka_1 + ka_2 + ka_3 + ka_4 + ka_5$ but

$\sum_{k=1}^{5} ka_k = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5$

4. $\sum_{i=m+1}^{n-1} f(i)$

5. a) $\sum_{i=0}^{5} ar^i = a + ar + ar^2 + ar^3 + ar^4 + ar^5$

*b) $\sum_{i=0}^{n-1} ar^i = a + ar + ar^2 + \dots + ar^{n-1}$

$r \sum_{i=0}^{n-1} ar^i = \quad ar + ar^2 + \dots + ar^{n-1} + ar^n$

Subtracting the first row from the second, we get:

$r \sum_{i=0}^{n-1} ar^i - \sum_{i=0}^{n-1} ar^i = -a + ar^n$

$(r - 1) \sum_{i=0}^{n-1} ar^i = a(r^n - 1)$

$\sum_{i=0}^{n-1} ar^i = \frac{a(r^n - 1)}{r - 1}$

## 4-2. Addition, Subtraction, Division, Multiplication and Composition of Functions

This section shows how functions may be combined to form new functions. The algebra of polynomials is developed from the functional point of view.

Rational functions are mentioned only briefly (see page 158) because of the problem of discontinuity which is not discussed in this text. A rational function is continuous at all points except when the denominator is zero. But since the domain does not include these points we say that a rational function is continuous at all points of its domain.

The comprehensive treatment of composition of functions is basic to much of the remaining work of the course. This discussion could logically be placed at the end of Chapter 1. The decision to postpone the treatment until this time was somewhat arbitrary. The teacher should feel free to introduce the work on composition of functions at any time following Chapter 1.

In the illustration under consideration on page 164 (top of page) the function $g \circ f$ is not necessarily the same as $f \circ g$.

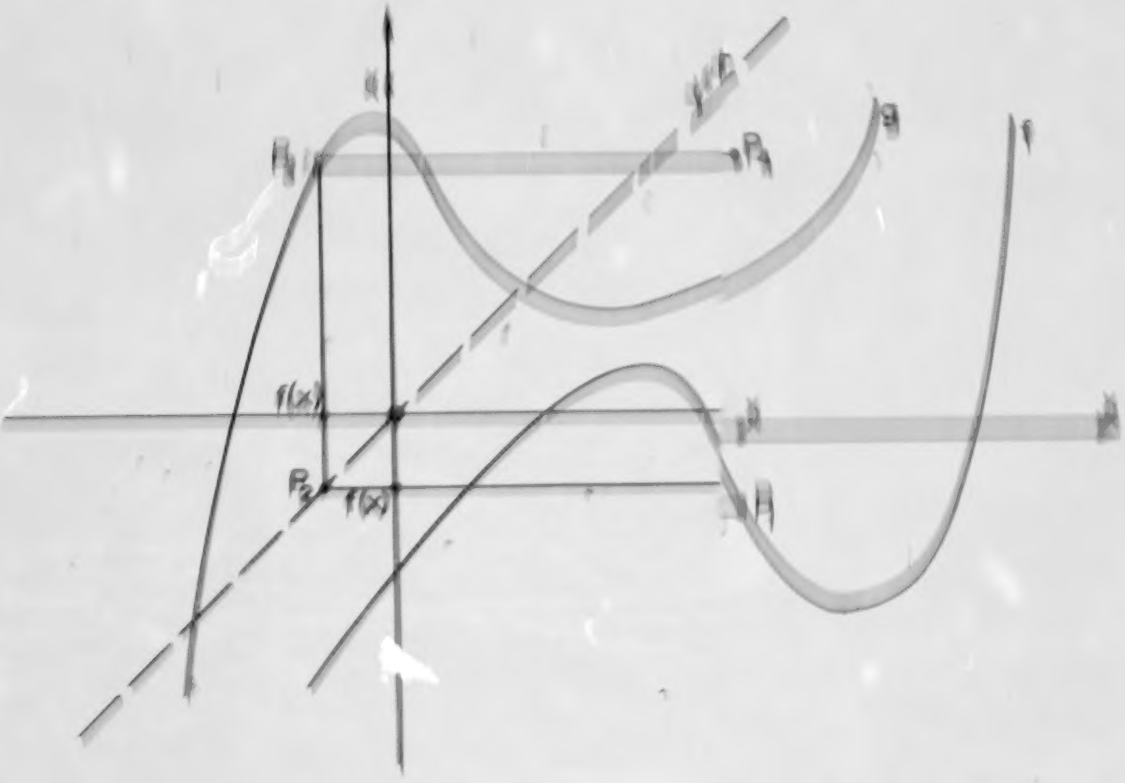


Figure TC3-2a. Point $P_4$ is on the graph of $g \circ f$.

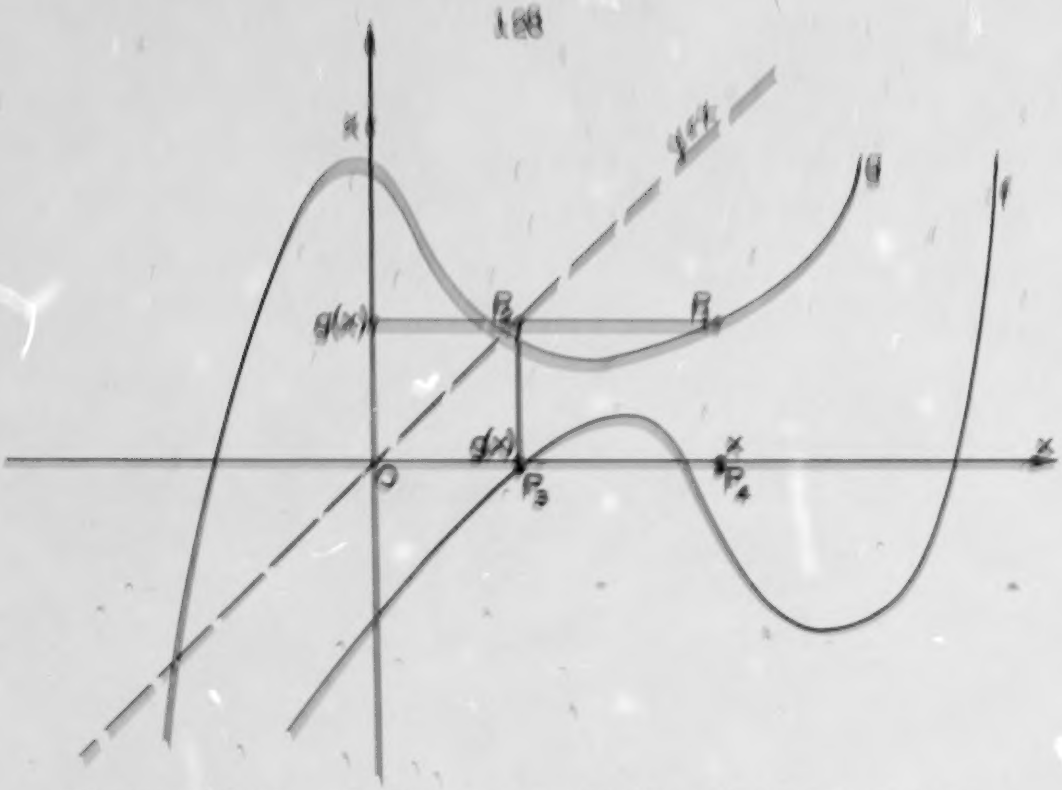


Figure TC3-2b. Point $P_4$ is on the graph of f o g.

In fact, the composition of functions is not generally commutative. A graphical analysis of g o f and f o g will clarify this statement. (See Figures TC3-2a and b.) To graph the g o f given the graphs of f and g we locate the points $P_1$, $P_2$, $P_3$, $P_4$ in sequential order where $P_1(x, f(x))$ is any point on the graph of f; $P_2(f(x), f(x))$ is the intersection of the line $y = x$ and the line through P which is parallel to the x-axis; $P_3(f(x), g(x))$ is the intersection of the graph of g and the line through $P_2$ which is parallel to the y-axis; and finally $P_4(x, g(f(x)))$ is the intersection of the line through $P_3$ which is parallel to the x-axis and the line through P which is parallel to the y-axis. The point $P_4$ is on the graph of g o f;

other points on the curve may be similarly obtained.

To sketch the graph of $f \circ g$ we take $P_1(x, g(x))$ any point on the graph of $g$ and successively locate the points

$$P_2(g(x), g(x)),$$
$$P_3(g(x), f(g(x))),$$
$$\text{and } P_4(x, f(g(x))).$$

The point $(x, f(g(x)))$ is on the graph of $f \circ g$.

Figures TC3-2a and b clearly show that the composition of functions is not in general commutative since $g(f(x)) \neq f(g(x))$.

## Graphing a function.

As we have seen in several cases, the study of the properties of a function is often helped by a consideration of its graph.

If $f(-x) = -f(x)$, the function is said to be an "odd" function and its graph is symmetric with respect to the origin. If $f(-x) = f(x)$, the function is called an "even" function and its graph is symmetric with respect to the $f(x)$ axis. The graph of a polynomial function cannot be symmetric with respect to the $x$-axis (except for the trivial case of the zero polynomial).

**Answers to Exercises 3-2a.** Page 157

1. When $(a_i + b_i) = 0$, for all $i$ from 1 to $n$ and $a_0 + b_0 \neq 0$, we will have a polynomial of degree less than 1. (For difference, change signs of each $b_i$.) If $a_0 + b_0$ also equals zero, the sum will not have a degree.

**Answers to Exercises 3-2b.** Page 158

1. a) $P(x) + Q(x) = x^4 - 6x$

   b) $P(x) - Q(x) = -x^4 + 4x^3 + 4x + 6$

   c) $P(x) \cdot Q(x) = 2x^7 - 4x^6 - x^5 - 5x^4 - 12x^3 + 5x^2 - 12x - 9$

2. a) $P(x) + Q(x) = 5x^3 - x^2 + 3x + 3$

   b) $3P(x) + 4Q(x) = 15x^3 - 2x^2 + 12x + 11$

   c) $P^2(x) - Q(x) = -25x^6 - 20x^5 + 4x^4 + 10x^3 - 5x^2 - 3x - 1$

3. a) $P(x) \cdot Q(x) = x^8 - 2x^7 - 4x^6 + 5x^5 + 12x^4 + 8x^3 - 32x^2 + 4x + 8$

   b) $P(x) + 2Q(x) = 3x^4 - 2x^3 - 4x^2 + 2x + 8$

   c) $P(x) - Q(x) = -2x^3 - 4x^2 + 11x + 8$

* 4. a) The addition of polynomials is commutative and associative.

   b) The multiplication of polynomials is commutative and associative. It is also distributive with respect to addition. Under addition and multiplication the real (or complex) numbers obey these laws. The rules for adding and multiplying polynomials are such that these laws go over to polynomial addition and multiplication.

Answers to Exercises 3-2c. Page 167

1. a) $(fg)(-2) = f(g(-2)) = f(0) = -1$

   b) $(gf)(0) = g(f(0)) = g(-1) = +1$

   c) $(gg)(1) = g(g(1)) = g(3) = +5$

   d) $(ffg)(1) = (ff)(3) = f(8) = +63$

   e) $(fg)(x) = f(x + 2) = x^2 + 4x + 3$

   f) $(gf)(x) = g(x^2 - 1) = x^2 + 1$

   g) $$\frac{(fg)(x) - (fg)(1)}{x - 1} = \frac{f(x + 2) - f(3)}{x \ \ x - 1} = \frac{x^2 + 4x + 3 - 8}{x - 1}$$

   $$= \frac{x^2 + 4x - 5}{x - 1} = x + 5 , \quad (x \neq 1)$$

2. a) $(fg)(x) = f(cx + d) = acx + ad + b$

   b) $(gf)(x) = g(ax + b) = acx + bc + d$

   c) Slope of $(fg)$ = Slope of $(gf)$ = $ac$ = product of slopes of $f$ and $g$.

   d) Theorem: $(fg)' = (gf)' = g' \cdot f' = f' \cdot g'$

3. a) $(f \circ f)(1) = 1$ ; $(f \circ f)(-3) = -3$ ;
   $(f \circ f)(8) = 8$

   b) If $f : x \longrightarrow \frac{1}{x}$ then $(f \circ f) : x \longrightarrow x$.

4. a) $(fj)$ and $(jf)$ are both equivalent to $f : x \longrightarrow x + 2$

   b) $g : x \longrightarrow x - 2$

   c) $h : x \longrightarrow x - 2$. The two are the same.

5. a) $(f \circ g)(x) = x^6$ and $(g \circ f)(x) = x^6$

   b) $(f \circ g)(x) = (g \circ f)(x) = x^{mn}$

6. a) $(f \cdot g)(x) = x^5$

   b) $(f \cdot g)(x) = x^{m+n}$

   The $\cdot$ operation is not the same as $\circ$ operation.

7. a) $(f \cdot g)(x) = x^2 - x - 6$

   b) $[(f \cdot g) \circ h(x)] = x^4 - x^2 - 6$

   c) $(f \circ h)(x) = x^2 + 2$

   d) $(g \circ h)(x) = x^2 - 3$

   e) $[(f \circ h) \cdot (g \circ h)](x) = x^4 - x^2 - 6$

8. Given 3 real functions $f : x \rightarrow f(x)$, $g : x \rightarrow g(x)$ and $h : x \rightarrow h(x)$, we wish to show that

$$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h).$$

[It is assumed throughout that f, g, and h are being discussed for all x in their common domain.]

By definition : $(f \cdot g)(x) = f(x) \cdot g(x)$

$$(f \cdot g) \circ h(x) = f(h(x)) \cdot g(h(x))$$
$$= (f \circ h) \cdot (g \circ h)$$

9. $f \circ (g \cdot h) = (f \circ g) \cdot (f \circ h)$ since

$$f(x) \circ (g(x) \cdot h(x)) = f(g(x) \cdot h(x)) = f(g(x)) \cdot f(h(x))$$
$$= (f \circ g) \cdot (f \circ h).$$

10. $(f + g) \circ h = (f \circ h) + (g \circ h)$ since $(f + g)(x) = f(x) + g(x)$, and if $x$ is replaced by $h(x)$, we obtain $(f + g)(h(x)) = f(h(x)) + g(h(x))$.

But $f \circ (g + h)$ does not equal $(f \circ g) + (f \circ h)$ since a counter example may be shown to be

$f : x \rightarrow ax + 1$

$g : x \rightarrow bx$

$h : x \rightarrow cx$

$$[f \circ (g + h)](x) = a(bx + cx) + 1 = abx + acx + 1$$

$$[(f \circ g) + (f \circ h)](x) = [a(bx) + 1] + [a(cx) + 1]$$

$$= abx + acx + 2$$

## 3-3. Evaluation of $P(x)$ at $x = c$.

Synthetic division is presented as an application of the distributive principle which is efficient and easily justified. The process here developed is used in computing with machines since the only operations required are successive multiplications and additions rather than raising to powers.

In Chapter 2 (Sections 2-2 and 2-4) we had occassion to expand a polynomial $f$ in the form

$$f(x) = b + b_1(x - h) + b_2(x - h)^2 + \dots + b_n(x - h)^n . \qquad (1)$$

Instead of following the procedure explained there it is possible to find the $b$'s (in the above expression) as remainders in divisions by $x - h$.

To obtain expansion (1) we first divide $f(x)$ by $(x - h)$ using synthetic division, obtaining the quotient $Q(x)$ and the remainder $R = f(h)$ so that

$$f(x) = f(h) + (x - h)Q(x). \qquad (2)$$

But since $Q(x)$ is also a polynomial we may write

$$Q(x) = Q(h) + (x - h)Q_1(x) \qquad (3)$$

where $Q_1(x)$ is the new quotient. Thus substitution of (3) in equation (2) gives

$$f(x) = f(h) + (x - h)Q(h) + (x - h)^2 Q_1(x).$$

Hence $b_0 = f(h)$ and $b_1 = Q(h)$. By continuing this process we obtain

$$f(x) = f(h) + (x - h)Q(h) + (x - h)^2 Q_1(h) + (x - h)^3 Q_2(x)$$

and so on. The required $b_2$ is given by $Q_1(h)$. The coefficients $f(h)$, $Q(h)$, $Q_1(h)$, ... (that is, $b_0, b_1, b_2, \ldots$) may be found quickly by successive synthetic divisions. This shorter method is now at our disposal. The teacher may wish to develop this simplified procedure as an alternative method. For example, to expand $f(x) = x^3 - 3x^2 + 4x + 2$ in powers of $x - 2$ we write

$$\begin{array}{cccc|l}
1 & -3 & +4 & +2 & \underline{2} \\
 & 2 & -2 & 4 & \\ \hline
1 & -1 & 2 & & 6 = f(2) = b_0 \\
 & 2 & 2 & & \\ \hline
1 & 1 & & & 4 = Q(2) = b_1 \\
 & 2 & & & \\ \hline
1 & & & & 3 = Q_1(2) = b_2 \\ \hline
\end{array}$$

$$1 = Q_2(2) = b_3 .$$

Hence

$$f(x) = 6 + 4(x - 2) + 3(x - 2)^2 + (x - 2)^3 ,$$

Answers to Exercises 3-3. page 179

1.

$$\begin{array}{cccc|c|c|}
1 & 0 & 0 & +1 & -3 & \\ \hline
1 & -2 & 4 & -7 & +11 & -2 \\
1 & 1 & 1 & 2 & -1 & 1 \\
1 & 3 & 9 & 28 & 81 & 3 \\ \hline
\end{array}$$

| | | | | | |
|---|---|---|---|---|---|
| 2. | -3 | +1 | +1 | -2 | |
| | -3 | +4 | -3 | 1 | -1 |
| | -3 | +10 | -29 | 85 | -3 |
| | -3 | +1 | +1 | -2 | 0 |
| | -3 | -5 | -9 | -20 | 2 |
| | -3 | -11 | -43 | -174 | 4 |

| | | | | | |
|---|---|---|---|---|---|
| 3. | 3 | -2 | +0 | +1 | |
| | 3 | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $\frac{7}{8}$ | $\frac{1}{2}$ |
| | 3 | -1 | $-\frac{1}{3}$ | $\frac{8}{9}$ | $\frac{1}{3}$ |
| | 3 | 4 | 8 | 17 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| 4. | 6 | -5 | -17 | +6 | |
| | 6 | +4 | -11 | $-\frac{21}{2}$ | $\frac{3}{2}$ |
| | 6 | -2 | -18 | -3 | $\frac{1}{2}$ |
| | 6 | -8 | -13 | $\frac{25}{2}$ | $-\frac{1}{2}$ |
| | 6 | -3 | -18 | 0 | $\frac{1}{3}$ |
| | 6 | +7 | -3 | 0 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| 5. | 6 | -29 | +37 | -12 | |
| | 6 | -29 | +37 | -12 | 0 |
| | 6 | -23 | +14 | +2 | 1 |
| | 6 | -17 | +3 | -6 | 2 |
| | 6 | -11 | +4 | 0 | 3 |
| | 6 | -5 | +17 | 56 | 4 |

## 3-4. Remainder and Factor Theorems

In this section the remainder and factor theorems are developed as an outgrowth of the process of synthetic division. The remainder theorem is applied as a testing device.

### Answers to Exercises 3-4.

1. $6x^3 - 5x^2 - 17x + 6$ has $(x - 2)$ as a factor and $(x - \frac{1}{3})$ or $(3x - 1)$ as a factor and $6x^3 - 29x^2 + 37x - 12$ has $(x - 3)$ as a factor.

2.

| | | | | |
|---|---|---|---|---|
| 2 | +1 | -5 | +2 | |
| 2 | -3 | +1 | 0 | -2 |
| 2 | -1 | -4 | 6 | -1 |
| 2 | +1 | -5 | +2 | 0 |
| 2 | 3 | -2 | 0 | 1 |
| 2 | 5 | 5 | 12 | 2 |

$$P(x) = 2x^3 + x^2 - 5x + 2$$
$$= (x + 2)(x - 1)(2x - 1)$$

3.

| | | | | |
|---|---|---|---|---|
| 1 | +4 | +1 | -6 | |
| 1 | [illegible] | 22 | 60 | 3 |
| 1 | 6 | 13 | 20 | 2 |
| 1 | +5 | +6 | 0 | 1 |
| 1 | +4 | +1 | -6 | 0 |
| 1 | +3 | -2 | -4 | -1 |
| 1 | +2 | -3 | 0 | -2 |
| 1 | +1 | -2 | 0 | -3 |

$$P(x) = x^3 + 4x^2 + x - 6$$
$$= (x - 1)(x + 2)(x + 3)$$

4.

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | -17 | -16 | +12 | |
| 2 | 7 | +4 | -4 | 0 | 3 |
| 2 | 5 | -7 | -30 | -48 | 2 |
| 2 | 7 | 4 | -4 | | |
| 2 | 3 | -2 | 0 | -2 | |

a) $P(3) = 0$

b) $P(2) = -48$

c) $P(x) = (x - 3)$
$(x + 2)^2(2x - 1)$

5. If $P(x) = x^3 + 3x^2 - 12x - K$

$$\begin{array}{rrrr|l} 1 & +3 & -12 & -K & \\ \hline 1 & +6 & +6 & 9 & 3 \end{array} \qquad (3 \cdot 6) - K = 9$$

$$K = 9$$

6\.

$$\begin{array}{rrrrrr|l} A & +A & +13 & -11 & -10 & -2A & \\ \hline A & 0 & 13 & -24 & 14 & (-2A - 14) & -1 \end{array} \qquad -2A - 14 = 0$$

$$2A = -14, \quad A = -7$$

$f(1) = -7 \ -7 \ +13 \ -11 \ -10 \ +14 = -8$

## 3-5. Zeros of Polynomials and Roots of Polynomial Equations

Zeros of a function are looked upon as solutions of the equation $f(x) = 0$, or alternately as the first component of the ordered pair $(x, f(x))$ where $f(x) = 0$. In succeeding sections the discussion is in terms of zeros of functions as roots of $f(x) = 0$.

### Answers to Exercises 3-5 Page 182

1. $-3 < x_A < -2$ ; $2 < x_B < 3$.

2. a) $P(x) = 2x^3 - 3x^2 - 12x + 13$

$P(1) = 2(1)^3 - 3(1)^2 - 12(1) + 13 = 2 - 3 - 12 + 13$

$= 0$,

b) $P(x) = (x - 1)(2x^2 - x - 13)$

c) If $P(x) = 0$ when $x \neq 1$, then since $x - 1 \neq 0$, $2x^2 - x - 13$ must equal $0$ because for real or complex numbers, the product of two factors can equal zero only if at least one of the factors is zero. Furthermore if $Q(x) = 0$, then

$P(x) = (x - 1) \cdot Q(x) = (x - 1) \cdot 0 = 0$ and so $x$ is a zero of $P$.

d) $x = 1$ ; $x = \frac{1 + \sqrt{105}}{4}$ and $x = \frac{1 - \sqrt{105}}{4}$

3. a) $(0, -4)$ and $(1, -5)$

   b) $(0, -4)$

   c) $(2, 0)$

   d) $(2, 0)$

   e) $(\frac{3}{2}, -4)$

   f) $(-\frac{1}{2}, -5)$

   g) $2 < x < 3$ or $2.1 < x < 2.25$.

## 3-6. Rational Zeros

The student should be encouraged to make intelligent guesses at the zeros of a function. At this time he should use all of the resources at his command.

The references listed at the close of this section provide a source of material for students of various ability levels.

### Answers to Exercises 3-6a. Page 190

1. a) $x = 2$, $x = -\frac{1}{2}$

   b) $x = 2$, $x = -\frac{1}{2}$ and $x = 0$

2. a) $x = -2$, $x = \frac{3}{2}$

   b) $x = -2$, $x = \frac{3}{2}$ and $x = 0$

3. a) $x = 1$, $x = 2$, and $x = 3$

   b) $x = 1$, $x = 2$, $x = 3$ and $x = 0$

4. a) $x = -2$, $x = -1$, and $x = 1$

   b) $x = -2$, $x = -1$, $x = 0$ and $x = 1$

5. a) $x = -2$, $x = 1$, and $x = 2$

   b) $x = -2$, $x = 0$, $x = 1$ and $x = 2$

6. a) $x = -1$, $x = \frac{1}{2}$ and $x = 1$

   b) $x = -1$, $x = 0$, $x = \frac{1}{2}$ and $x = 1$

7. $x = -2$, $x = -1$, $x = 2$ and $x = 3$

8. $x = +2$, $x = -2$ (Each of these are double roots.)

9. $x = +3$, $x = +2$, $x = -1$, and $x = 1$

10. $x = -3$, $x = -2$, $x = -1$, $x = 1$ and $x = 2$

11. $x + \frac{1}{x} = N \Longrightarrow x^2 - Nx + 1 = 0$

    The discriminant for this quadratic is $N^2 - 4$.

    If $|N| < 2$, $N^2 < 4$ and $(N^2 - 4) < 0$ which means the roots are imaginary.

## 3-7. The Location Theorem

The statement of this theorem is made plausible on the basis of a graphical interpretation. Since no formal study of continuity has been included, an intuitive appeal is the only justification for the assumption that the polynomial function is continuous in any closed interval.

An intelligent guess at the zero of a polynomial and application of the Location Theorem will enable the student to calculate the zero to any required degree of accuracy. However, the procedure may lead to tedious, involved computation.

Answers to Exercises 3-7 Page 199

1. a) $x \approx 1.95^+$

   b) $x = -1$

   c) $x = -5$ ; Note that there can be NO positive roots.

   d) $x = 12.$

2. $x^3 + x = 3.$ To find $x$, solve $x^3 + x - 3 = 0$

   $f(1) < 0$; $f(2) > 0$; $f(1.5) > 0$; $f(1.25) > 0$ ;

   $f(1.2) < 0.$

   $x \approx 1.2^+$

3. a) $-1 < x < 0$ ; $1 < x < 2$ ; and $2 < x < 3$

   b) $0 < x < 1$

   c) $1 < x < 2$

   d) $-2 < x < -1$

   e) $-1 < x < 0$ ; $1 < x < 2$ ; and $2 < x < 3$

   f) $2 < x < 3$

   g) $-2 < x < -1$; $0 < x < 1$; $1 < x < 2$; and

      $5 < x < 6$

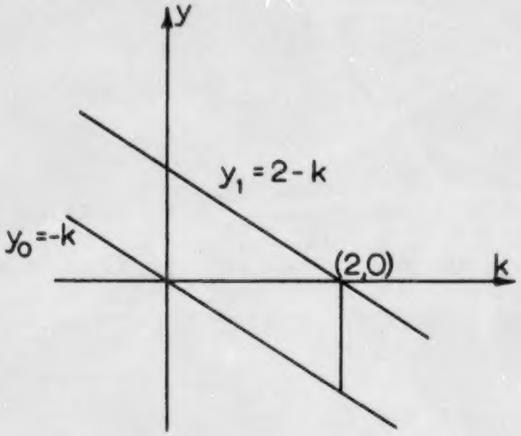
4. $x \approx 2.0^+$

5. Since $P(3) = -1$ and $P(4) = 13$, $x \approx 3.0^+$

6. $P(x) = x^3 - 2x^2 + 3x - k$.

a) $P(0) = -k$ and $P(1) = 2 - k$

For $-k$ and $2 - k$ to be different in sign, we must have $0 < k < 2$, since, if $k < 0$, both $-k > 0$ and $2 - k > 0$. If $k > 2$, both $-k < 0$ and $2 - k < 0$. Another way to see this is to graph $y_0 = -k$ and $y_1 = 2 - k$. When both $y_0$ and $y_1$ are positive or both negative we cannot use that value of $k$. But when one $y$ value is above the $k$-axis and the other is below, we have a possible $k$-value.



b) $P(1) = 2 - k$ and $P(2) = 6 - k$

$2 < k < 6$.

## 3-8. Newton's Method

The objective of sections 3-6 through 3-8 is to develop theoretical and practical methods for finding zeros of functions. It is possible to obtain an approximation to the zero(s) of a function for inspection of the graph. Horner's method is sometimes used to determine zeros of a polynomial to any desired degree of accuracy. However in this text Newton's method has been selected for several reasons:

1. It illustrates an application of the slope function discussed in Chapter 2.

2. It has the advantage of being applicable to the solution of transcendental as well as algebraic equations.

3. It usually converges to the root very rapidly. In fact, the degree of accuracy is doubled with each application.

It should be noted that there are situations in which Newton's method fails to converge. We have pointed out some of these in the next few pages.

A good first guess (or approximation for the root) decreases the error involved in using Newton's method. In order to obtain a satisfactory first approximation it is often desirable to use "straight line interpolation." Thus, if the function changes sign between two successive integers $n_1$ and $n_2$, we approximate the zero of the function by $x_1$ the value of $x$ at the point where the straight line through $(n_1, p(n_2))$ and $(n_2, p(n_2))$ crosses the x-axis. (See Figure TC 3-8a.)

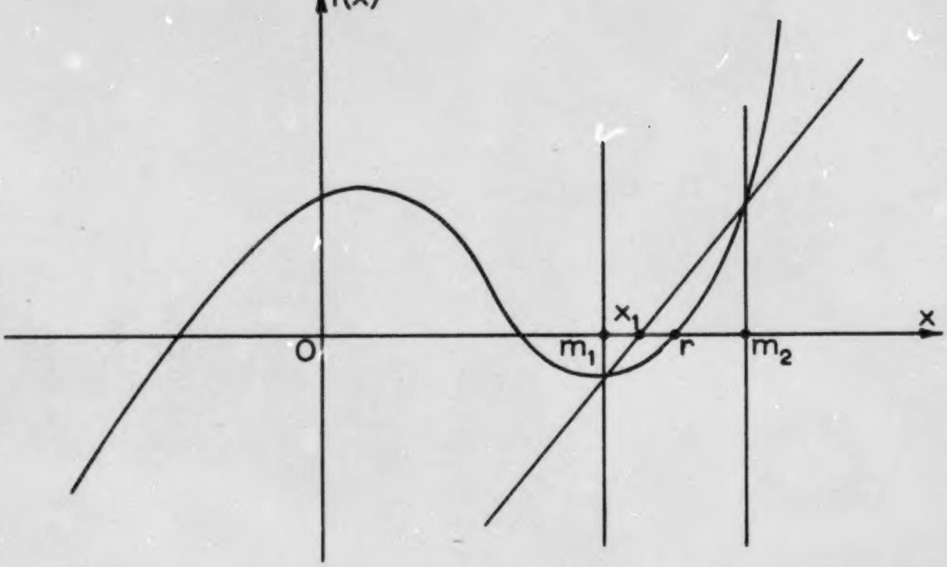


Figure TC 3-8a. Using straight line interpolation to locate the root r.

It is readily seen that this approximation is too small if the curve P is concave upward in the interval $[n_1, n_2]$, and too large if the curve is concave downward in this interval.

To illustrate the procedure we use straight line interpolation to approximate that zero of the function

$$f: \ x \longrightarrow 2^x - 2x^3 + 3x^2 + 12x - 1$$

which is the interval (3, 4). (Note. This exercise is taken from Chapter 4, Miscellaneous Exercises, Ex.35).

If $x = 3$, the value of the function is 16 and if $x = 4$, the value of the function is -17. Hence there is a root in the interval (3, 4). (Refer to Figure TC 3-8b).

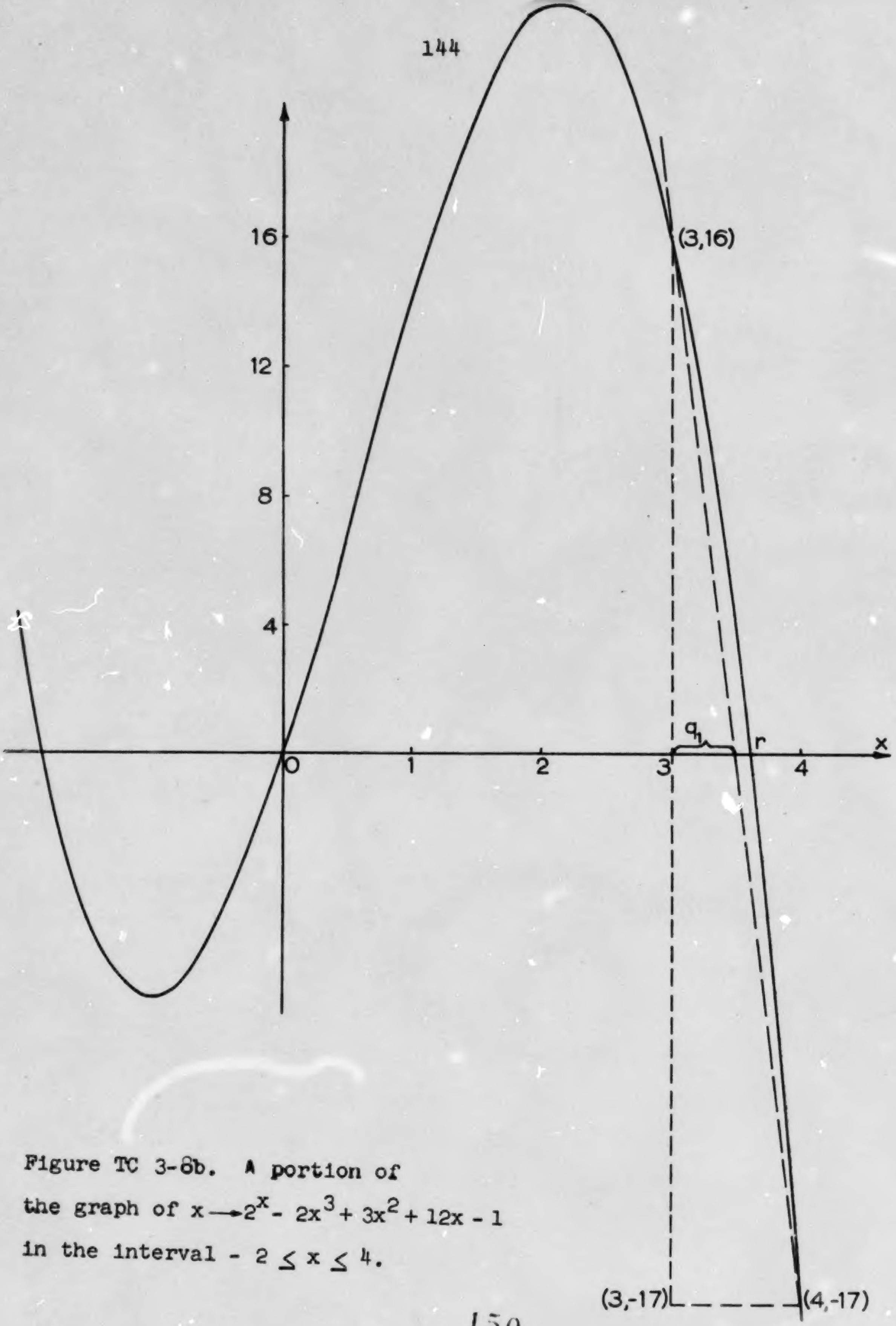


Figure TC 3-8b. A portion of the graph of $x \longrightarrow 2^x - 2x^3 + 3x^2 + 12x - 1$ in the interval $-2 \leq x \leq 4$.

Since by similar triangles $q_1/1 = 16/33 \approx 0.5$, our first approximation is $3 + 0.5 = 3.5$. We may use straight line interpolation again (See Figure TC 3-8c.) to obtain $q_2$. Thus $q_2/0.1 \approx 3.3/3.4 \approx 0.97$ or $q_2 \approx 0.10$ and our second straight line approximation is $3.5 + 0.1 = 3.6$ to the nearest one-tenth.

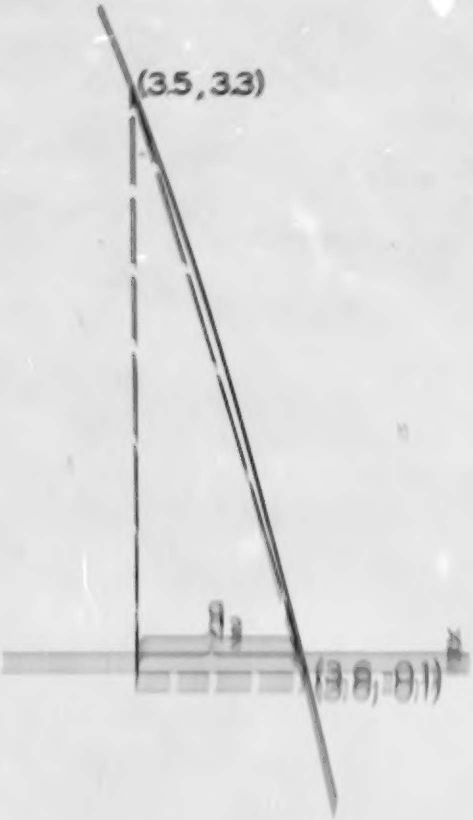


Figure TC 3-8c. A portion of the graph of $y = -x^4 - 2x^3 + 3x^2 + 12x - 1$ in the interval $3.5 \leq x \leq 3.6$.

In a favorable situation with a good first approximation Newton's process works for polynomials and for other continuous functions. Under certain circumstances the method may fail to converge. For example, if the tangent at $(x_1, p(x_1))$ meets the x-axis at $x_2$ and the tangent at $(x_2, p(x_2))$ meets the x-axis at $x_1$, successive approximations by Newton's method fluctuate back and forth between $x_1$ and $x_2$. (See Example in Figure TC 3-8d.)

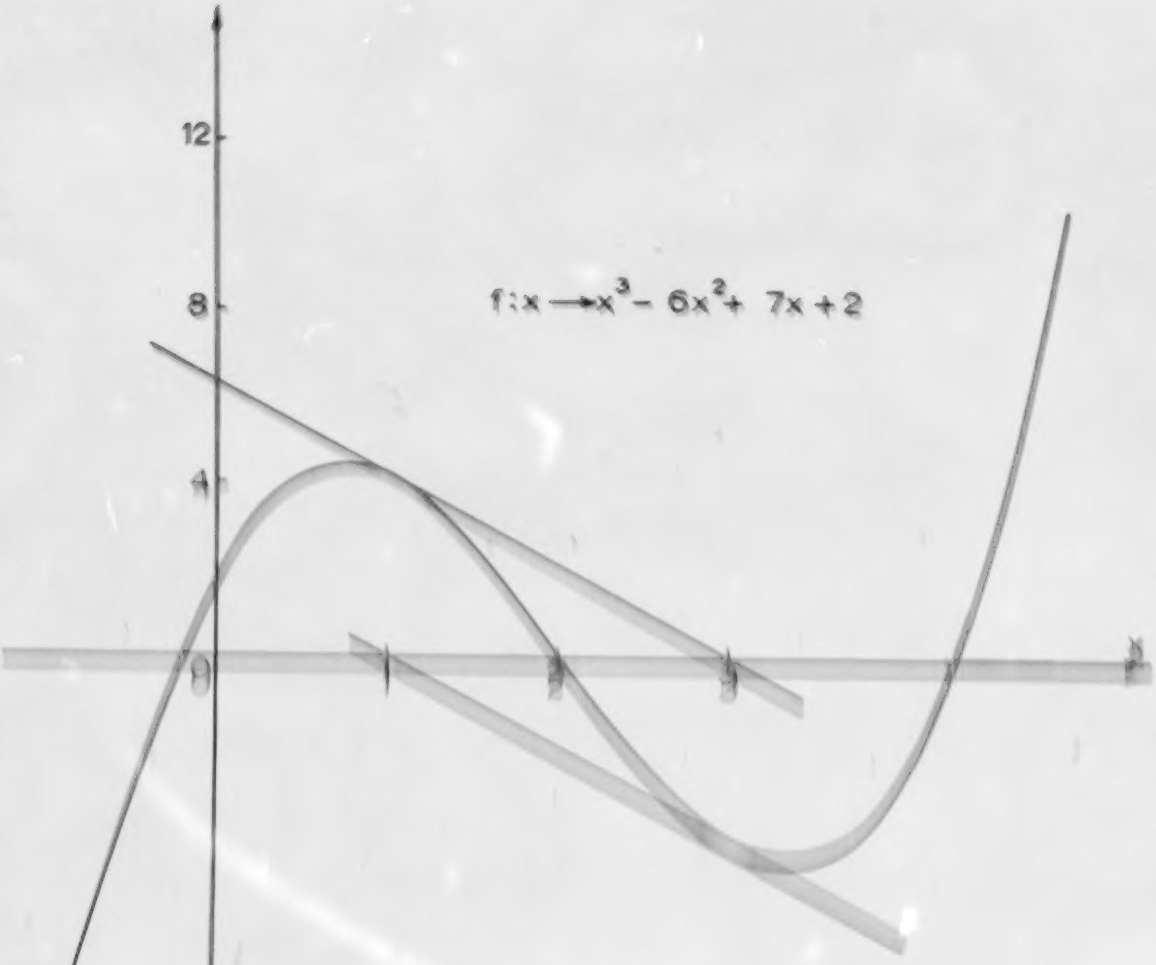


Figure TC 3-8d. Successive approximations by Newton's method fluctuate back and forth between $x_1 = 1$ and $x_2 = 3$.

In this case it is apparent that the graph of the polynomial has a point of inflection between $x_1$ and $x_2$. Successive approximations by Newton's method may actually get worse. This is illustrated in Figure TC 3-8e.

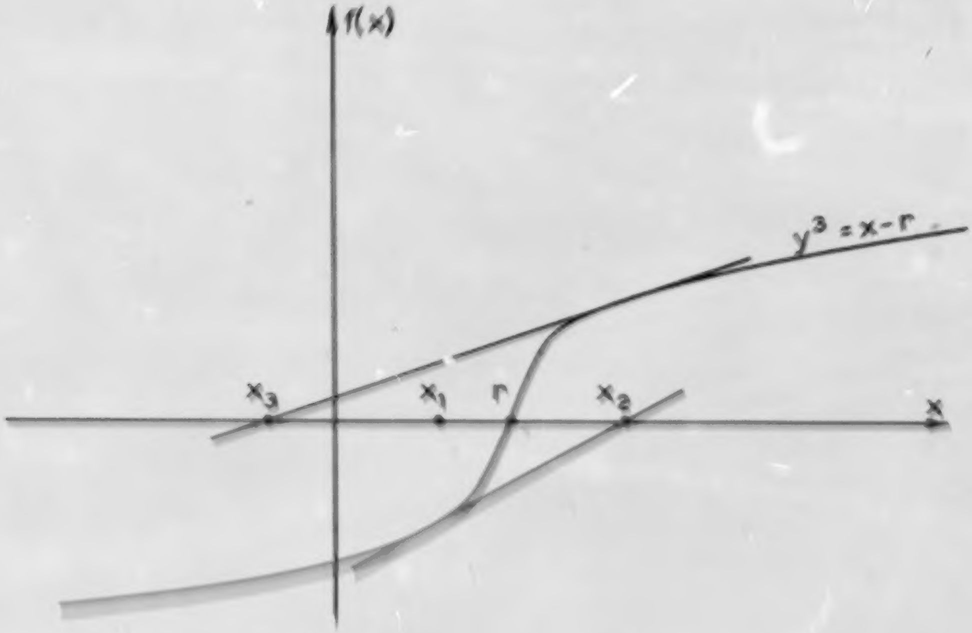


Figure TC 3-8e. Successive approximations by Newton's method may actually get worse.

Here again it should be noted that the curve has a point of inflection in the interval $[x_1, x_2]$. In such a case straight line interpolation may be used to give a satisfactory first approximation to the root $r$.

For further discussion of the error involved the teacher may refer to George B. Thomas, Elements of Calculus and Analytical Geometry, Addison-Wesley Publishing Co., Inc. 1959, pages 165-171.

**Answers to Exercises 3-8** Pages 204-205

1. $f: x \longrightarrow x^2 - 2 = 0$; $f'(x) = 2x$.

$x_{n+1} = \dfrac{-f(x_n)}{f'(x_n)} + x_n$ The first choice of $x$ should probably be $x_1 = 1.5$.

$x_2 = \dfrac{-f(1.5)}{f'(1.5)} + 1.5 = \dfrac{-.25}{3} + 1.5 = -.0833 + 1.5 = 1.4167$

$x_3 = \dfrac{-.00703889}{2.8334} + 1.4167 = 1.4142$ which checks with table value of $\sqrt{2} = 1.414$

2. $x^2 + x - 4 = 0$ by formula

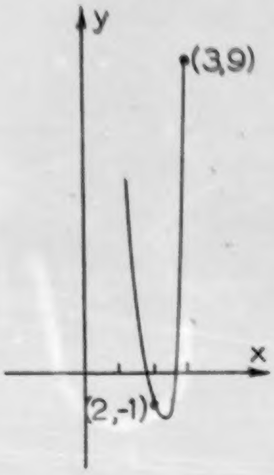
$x_1 \approx 1.562$ $x_1 = \dfrac{-1 + \sqrt{17}}{2} = \dfrac{-1 + 4.123}{2} = 1.5615$

$x_2 \approx -2.562$ $x_2 = \dfrac{-1 - \sqrt{17}}{2} = \dfrac{-1 - 4.123}{2} = -2.5615$

3. $f(x) = x^3 - 12x + 1$

$x \approx 0.083$ for $f(x) = 0$ $\begin{cases} f(0.083) = .004+ \\ f(0.084) = -.0074 \end{cases}$

4. 2.1 is much closer to the secant intersection.



## 3-9. Complex Zeros of Polynomials; Number of Zeros

In this section the treatment of zeros of polynomials is extended to the case where the coefficients of the polynomials are complex numbers (which may be real). The fundamental theorem of algebra (that every algebraic equation whose coefficients are complex numbers has a solution which is a complex number) is informally developed, illustrated and discussed.

The set of zeros of $P_n$ is the solution set of the equation $P(x) = 0$. If we set $P_n(x) = 0$, the solution set is $[r_1, r_2, \cdots, r_k]$ where $k = n$ if the multiplicity of each zero is 1, and $k < n$ if the multiplicity of any zero (that is, of any $r_1$) is greater than 1.

The agreement to count the roots of $P_n(x) = 0$ (or the zeros of $P_n(x)$) in the manner stated in the text is purely a matter of convenience. It enables us to say simply that any equation of degree $n$ has $n$ roots. Strictly speaking we mean that $P_n(x)$ may be written as the product of $n$ linear factors (not all of which need be different). From the point of view of set-theory, the solution set for $P_n(x) = 0$ may very well contain less than $n$ elements. This notation should cause no confusion since the sum of the multiplicities of the zeros of $P_n$ is $n$.

## Answers to Exercises 3-9a

1. $[x - (2+i)][x-(2-i)][x-1][x-(3-2i)][x-(3+2i)]$
$= (x^2 - 4x + 5)(x - 1)(x^2 - 6x + 13)$
$= (x^4 - 10x^3 + 42x^2 - 32x + 65)(x - 1)$
$= x^5 - 11x^4 + 52x^3 - 124x^2 + 147x - 65$

The coefficient of $x^4$ is $-11$. The sum of the zeros is $(2 + i) + (2 - i) + 1 + (3 - 2i) + (3 + 2i)$ which is also $-11$.

The constant term is $-65$. The product of the zeros is $(2 + i)(2 - i)(1)(3 - 2i)(3 + 2i)$ which is 65. The product is the negative of the constant term.

2. a) $x - (2 + 3i) = x - 2 - 3i$.
   b) $[x - (2 + 3i)][x - (2 - 3i)] = [(x -2) -3i][(x-2) + 3i]$
   $= x^2 - 4x + 13$

3. a) $x = 1$, $x = \frac{-1 + \sqrt{-3}}{2}$ or $\frac{-1 + i\sqrt{3}}{2}$, $x = \frac{-1 - \sqrt{-3}}{2}$ or $\frac{-1 - i\sqrt{3}}{2}$.
   b) $x = -1$, $x = \frac{+1 + \sqrt{-3}}{2}$ or $\frac{+1 + i\sqrt{3}}{2}$, $x = \frac{+1 - \sqrt{-3}}{2}$ or $\frac{+1 - i\sqrt{3}}{2}$.
   c) $x = 2$, $x = \frac{-1 + \sqrt{-15}}{2}$ or $\frac{-1 + i\sqrt{15}}{2}$, $x = \frac{-1 - \sqrt{-15}}{2}$ or $\frac{-1 - i\sqrt{15}}{2}$.
   d) $x = +i$, $x = -i$, $x = 2i$, $x = -2i$.
   e) $x = 1$ (double root), $x = +3i$, $x = -3i$.

4. Using either of the proofs of Theorem 3-7 as a model, two proofs may be shown. Only one is given here.

Given: $P(a + b\sqrt{2}) = 0$ $\quad$ a and b are rational numbers, $b \neq 0$.

Prove: $P(a - b\sqrt{2}) = 0$

Proof: Let $S(x) = [x - (a + b\sqrt{2})][x - (a - b\sqrt{2})]$

$$= [(x - a) - b\sqrt{2}][(x - a) + b\sqrt{2}]$$

$$= (x - a)^2 - 2b^2.$$

The coefficients of $S(x)$ are rational. If $P(x)$ is divided by $S(x)$ we get a quotient $Q(x)$ and a remainder $R(x) = hx + k$, possibly of degree 1 (but no greater) where h, k and all coefficients of Q are rational. Thus,

$$P(x) = S(x) \cdot Q(x) + hx + k.$$

This is an identity in x. By hypothesis, $P(a + b\sqrt{2}) = 0$ and from $S(x)$ above $S(a + b\sqrt{2}) = 0$, so we get

$$0 = 0 + ha + hb\sqrt{2} + k.$$

If $hb\sqrt{2}$ is not zero we get

$\sqrt{2} = \dfrac{-ha - k}{hb}$ where h, a, k and b are rational, which is impossible.

So $hb\sqrt{2} = 0$, and since $b \neq 0$, h must equal zero and and as a consequence k must equal zero. Therefore

$$P(x) = S(x) \cdot (Q(x).$$

Since $S(a - b\sqrt{2}) = 0$, it follows that

$P(a - b\sqrt{2} = 0.$ q.e.d.

5. $P(x) = [x - (3 + 2\sqrt{2})][x - (3 - 2\sqrt{2})] = x^2 - 6x + 1.$

6. For $a + b\sqrt{3}$, the proof given in answer to Exercise 4, with $\sqrt{3}$ substituted for $\sqrt{2}$ will be correct.

For $a + b\sqrt{4}$ there is no comparable theorem since the root is rational and there is no conjugate surd. If a proof like that in Exercise 4 is attempted it breaks down because at the step

$$\sqrt{2} = \frac{-ha - k}{hb}$$

if 4 is substituted for 2 both sides are rational and the contradiction needed in the proof does not appear.

Answers to Exercises 3-9b Pages 215 - 216

1. a and b are unstable.

c, d and e are stable.

2. a) $x^3 - 3x - 2 = 0$; $x = -1$, double root and $x = 2$.

3rd degree -- 3 roots.

b) $x^3 - 3x + 2 = 0$; $x = 1$, double root and $x = -2$.

3rd degree -- 3 roots.

c) $x^4 + 5x^3 + 9x^2 + 7x + 2 = 0$; $x = -1$, triple root and $x = -2$.

4th degree -- 4 roots.

d) $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1 = 0$ $\left\{\begin{array}{l}\text{Double roots: } x = -1, \\ x = i,\ x = -i.\end{array}\right.$

6th degree -- 6 roots

e) $x^6 - 2x^5 + 3x^4 - 4x^3 + 3x^2 - 2x + 1 = 0$ $\left\{\begin{array}{l}\text{Double roots: } x = 1, \\ x = i,\ x = -i.\end{array}\right.$

6th degree -- 6 roots

3. We have seen exceptions to all but (e). For the complex numbers there is a theorem establishing the fact.

   A recitation of exceptions follows:

   a) $2x - 1 = 0$ has a root not an integer.
   b) $x^2 - 2x - 2 = 0$ has roots which are not rational.
   c) $x^2 + 1 = 0$ has imaginary roots.
   d) $ix + i = 0$ has a real root.

4. The students might state that the Fundamental Theorem of Algebra would require a complex zero for equations like $x^4 + 1 = 0$ and $x^6 + 1 = 0$. If the students have used De Moivre's Theorem to find roots of complex numbers, they might point out that roots of complex numbers are complex numbers. As a matter of fact, complex powers of complex numbers and complex roots of complex numbers lead to nothing more complicated than complex numbers. (See Fehr, Secondary Mathematics.)

Answers to Miscellaneous Exercises of Chapter 3    Pages 217-222.

1. The better student should recognize the quantity in parentheses as a zero of the general quadratic: $f: x \longrightarrow ax^2 + bx + c$. Therefore, when used in place of $x$ we should get zero.

*2. $(A + Bx) \sum_{i=1}^{n} a_i x^i = A \sum_{i=1}^{n} a_i x^i + Bx \sum_{i=1}^{n} a_i x^i =$

$$= A \sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{n} Ba_i x^{i+1} =$$

$$= Aa_1 x + \sum_{i=2}^{n} Aa_i x^i + \sum_{i=2}^{n} Ba_{i-1} x^i$$

$$+ Ba_n x^{n+1} =$$

$$+ Aa_1 x + Ba_n x^{n+1} + \sum_{i=2}^{n} x^i (Aa_i + Ba_{i-1}).$$

*3. If $\sum_{i=1}^{n} f(i) = n^2(n + 1)^2$, then using $n - 1$ for $n$, we have $\sum_{i=1}^{n-1} f(i) = (n-1)^2(n)^2$ and subtracting, we get

$$f(n) = n^2(n+1)^2 - n^2(n-1)^2 = n^2[(n+1)^2 - (n-1)^2] =$$

$$= n^2 \cdot [(n+1) + (n-1)][(n+1)-(n-1)]$$

$$= n^2[2n \cdot 2] = 4n^3.$$

$$f(n) = 4n^3.$$

*4. $\sum_{i=1}^{n} 4i^3 = n^2(n+1)^2$

$$4 \sum_{i=1}^{100} i^3 = (100)^2(101)^2 = 102,010,000$$

$$\sum_{i=1}^{100} i^3 = 25,502,500.$$

5. $P_4(x) = x^4 - 6x^2 - 5 = (x^2 - 3)^2 - 14$.

For real values of $x$, $(x^2 - 3)^2$ can never be negative. This means $P(x)$ has a minimum value of $-14$. $(x^2 - 3)^2$ can have any positive value, so $P(x)$ will have the range $P(x) \geq -14$. Another way, is to obtain the zeros for $P'(x)$. They are 0, $\sqrt{3}$, $-\sqrt{3}$. Zero turns out to be a relative maximum. $\sqrt{3}$ and $-\sqrt{3}$ produce the same minimum value of $-14$. Since this graph will look like the letter W no values will occur below $-14$.

*6. Analysis for Theorem 3-2. Denote the degree of the zero polynomial by k. Either P or Q or both may have degree k.

If both, we get $k + k = 2k = k$.

If one, we get $k + n = k$.

There is no real value of k which satisfies both of these equations. This combines with the result in the text to show why the zero polynomial does not have a degree assigned to it.

7. a) Either rn or rm whichever is largest will be the maximum degree.

   b) Either rn or sm whichever is the largest will be the maximum degree.

   c) $rn + sm$.

8. The degree of $R(x)$ would be $n - m$.

9. a) $P(0) = -1$.

   b) $P(1) = 3$.

   c) $P(2) = 45$.

10. a) $P(0) = 9$.

    b) $P(-3) = -84$.

    c) $P(\frac{1}{2}) = 7$.

11. a) $P(0) = -3$.

    b) $P(-2) = -31$.

    c) $P(\frac{1}{3}) = \frac{20}{9}$.

*12. a) $P(1) = -7 + i$.

    b) $P(i) = 0$.

    c) $P(1 + i) = -6 - 8i$.

    d) $P(1 - i) = -2 + 4i$.

    e) $P(-1 + i) = -6 + 8i$.

*13. Students assigned this problem should be warned not to expect a unique polynomial.

$P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

$P(0) = a_0 \qquad = -5$.

$P(1) = a_0 + a_1 + a_2 + a_3 \qquad = -5$.

$P(-1) = a_0 - a_1 + a_2 - a_3 \qquad = -11$.

Adding the last two, we get $2a_0 + 2a_2 = -16$, but

$a_0 = -5$ so $a_2 = -3$.

Subtracting in the above two, gives us

$2a_1 + 2a_3 = 6$.

So $a_1 = 3 - a_3$ or $a_3 = 3 - a_1$.

b) $P_3(x) = -5 - a_1x - 3x^2 + (3 - a_1)x^3$.

or $-5 - (3 - a_3)x - 3x^2 + a_3x^3$.

a) $P(2) = 7 - 10a_1$ or $-23 + 10a_3$.

14. a) $3x^3 + x^2 - 2x - 1 - \frac{4}{x-2}$.

b) $x^3 - x^2 + x - 1$.

c) $5x^2 + 4x - 2 - \frac{16}{x-3}$ .

15. a) $x^2 - x + 3 + \frac{4}{2x+1}$ .

b) $27x^3 - 18x^2 + 12x - 8 + \frac{32}{3x+2}$.

*16. a) $[(a - b)x^2 + a^2(b - x) + b^2(x - a)] \div (x - a)$

$=[(a - b)x^2 - x(a^2 - b^2) + (a^2b - ab^2] \div (x - a) =$

$=[(a - b)(x^2 - \{a + b\}x + ab)] \div (x - a) =$

$=[(a - b)(x - a)(x - b)] \div (x-a) = (x-b)(a-b)$

or $ax - bx - ab - b^2$.

b) $[x^3 - (a + b + c)x^2 + (bc + ac + ab)x - abc] \div (x-a)$

$[(x - a)(x - b)(x - c)] \div (x - a) = (x-b)(x-c)$ or

$x^2 - x(b + c) + bc$.

17. Since $P_n(a) = 0$ by hypothesis, $P_n(x)$ is exactly divisible by $(x - a)$ or

$P_n(x) = (x - a) \cdot Q(x)$.

If now $x = b$,

$P_n(b) = (b - a) \cdot Q(b)$ which is equal to 0.

Since $b \neq a$, $b - a \neq 0$, therefore $Q(b)$ must equal zero. If $Q(b) = 0$, it is divisible by $(x - b)$ without a remainder or

$Q(x) = (x - b) \cdot R(x)$ , and so

$P_n(x) = (x - a) \cdot (x - b) \cdot R(x)$.

*18. If $P_n(x) = x^n - a^n$ then $P_n(a) = a^n - a^n = 0$ which means $P_n(x)$ or $x^n - a^n$ is divisible by $(x - a)$. If $n$ is even, $P_n(-a) = (-a)^n - a^n = (-1)^n(a^n) - a^n$. Since $n$ is even, $(-1)^n = +1$ and $a^n - a^n = 0$.

$\therefore$ $P_n(x) = x^n - a^n$ is divisible by $(x + a)$ when $n$ is even.

19. $P_3(x) = K(x - 1)(x - 4)(x + 4) = K(x^3 - x^2 - 16x + 16)$

$P_3(2) = K(1)\,(-2)(6) = -12K = -16$ or $K = \frac{4}{3}$

$$P_3(x) = \frac{4x^3}{3} - \frac{4x^2}{3} - \frac{64x}{3} + \frac{64}{3}$$

*20. To prove: $P(x) \div (x - a)(x - b)$, $a \neq b$, will leave a remainder of $\dfrac{[P(a) - P(b)]x + aP(b) - bP(a)}{a - b}$

$$\begin{aligned}
P(x) &= (x - a)(x - b)Q(x) + ux + v \\
P(a) &= ua + v \\
P(b) &= ub + v \\
P(a) - P(b) &= u(a - b) \\
aP(b) &= uab + av \\
bP(a) &= uab + bv \\
aP(b) - bP(a) &= v(a - b)
\end{aligned}$$

$$u = \frac{P(a) - P(b)}{a - b} \qquad v = \frac{aP(b) - bP(a)}{a - b}$$

Hence, the remainder is

$$\frac{[P(a) - P(b)]x + aP(b) - bP(a)}{a - b}$$

*21. Here we have a situation in which an apparent simplification makes a problem more difficult. Thus, when $b = a$, the result of Problem 20 becomes meaningless and the method used therein breaks down (there being only one equation for the determination of $u$ and $v$). We shall proceed along other lines.

We have $P(x) = (x - a)Q_1(x) + P(a)$ and

$$Q_1(x) = (x - a)Q_2(x) + Q_1(a).$$ Combining these,

$$P(x) = (x - a)^2 Q_2(x) + (x - a) Q_1(a) + P(a),$$

and the required remainder is seen to be $xQ_1(a) + P(a)-aQ_1(a)$. To determine $Q_1(a)$ in terms of $P(x)$ and a we shall use a result of Section 2-5. Noting that

$$Q_1(x) = \frac{P(x) - P(a)}{x - a},$$ we seek the value at $x = a$

of the polynomial $Q_1(x)$ obtained by performing the indicated division. But this was shown to be $P'(a)$ where $P'(x)$ is the slope function of $P(x)$. That is, $Q_1(a) = P'(a)$ and the remainder we seek is $xP'(a) + P(a) - aP'(a)$.

22. $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$

$P(2) = 0 = a_0 + 2a_1 + 4a_2 + 8a_3.$

$P(3) = 0 = a_0 + 3a_1 + 9a_2 + 27a_3.$

$P(0) = 6 = a_0.$

$P(1) = 12 = a_0 + a_1 + a_2 + a_3.$

Since $a_0 = 6$, $\begin{cases} 2a_1 + 4a_2 + 8a_3 = -6 \\ 3a_1 + 9a_2 + 27a_3 = -6 \\ a_1 + a_2 + a_3 = 6 \end{cases}$

$$\begin{cases} 6a_2 + 24a_3 = -24 \\ 2a_2 - 6a_3 = -18 \end{cases}$$

$$6a_3 = 30$$

$$a_3 = 5$$

$$a_2 = -24$$

$$a_1 = 25$$

$$a_0 = 6$$

And $P(x) = 6 + 25x - 24x^2 + 5x^3$.

23. a) $P(x) = (x - 2)^2$.

Zeros are at $x = 2$ (Double root).

y-intercept is 4.

Relative minimum is (2,0).

No points of inflection.

As $|x|$ gets large, $P(x)$ behaves like $x^2$.

b) $P(x) = (2 - x)^3$.

Zeros are at $x = 2$ (Triple root).

y-intercept is at 8.

No relative minimum or maximum points.

Point of inflection is at (2,0).

As $|x|$ gets large, $P(x)$ behaves like $-x^3$.

c) $P(x) = (x - 2)^4$

Zeros are at $x = 2$ (4 roots).

y-intercept is 16.

A relative minimum occurs at (2,0).

No point of inflection.

For large $|x|$, $P(x)$ behaves like $x^4$.

d) $P(x) = (x - 1)^2(x + 2)$.

Zeros are at $x = -2$ and $x = 1$ (double root).

y-intercept is 2.

A relative minimum occurs at (1,0).

A relative maximum occurs at (-1,4).

A point of inflection is at (0,2).

As $|x|$ gets large , $P(x)$ behaves like $x^3$.

e) $P(x) = (x + 1)^2 (x - 2)^2$.

Zeros occur at $x = 2$ (double root) and $x = -1$ (double root).

y-intercept is 4.

Relative minima occur at (2,0) and (-1,0).

A relative maximum occurs at $(\frac{1}{2}, \frac{81}{16})$.

Two points of inflection are at $x = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$.

As $|x|$ gets large , $P(x)$ behaves like $x^4$.

f) $P(x) = x^3 - 6x^2 -11x - 6 = (x - 1)(x - 2)(x - 3)$.

Zeros occur at $x = 1$, $x = 2$, and $x = 3$.

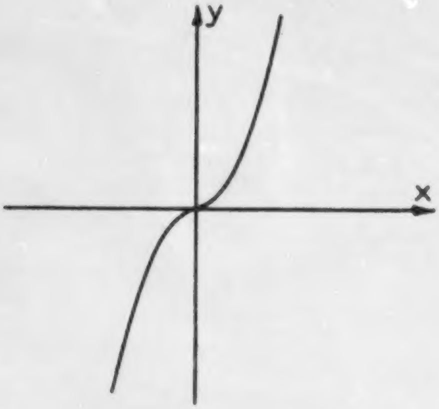
A relative maximum occurs at $\frac{6 - \sqrt{3}}{3}$

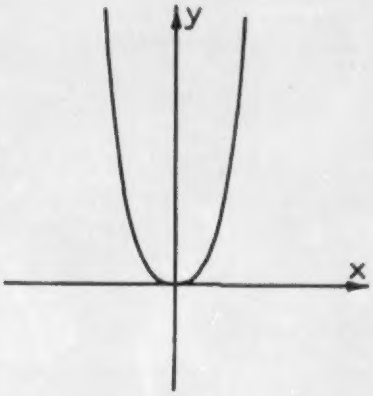
A relative minimum occurs at $\frac{6 + \sqrt{3}}{3}$

A point of inflection occurs at (2,0).

As $|x|$ gets large, $P(x)$ behaves like $x^3$.

*24. a) $f(x) = x|x|$. b) $f(x) = x^2\,|x|$.





25. a) $x^2(x + 3) = 4$.

$x^3 + 3x^2 - 4 = 0$.

$(x - 1)(x + 2)^2 = 0$ and $x = 1$, $x = -2$ (double root).

b) $(x + 1)(x + 2)(x + 3) = (x + 1)(x + 2)(x + 3)(x + 4)$.

$$[(x + 1)(x + 2)(x + 3)] - [(x + 1)(x + 2)(x + 3)(x + 4)] = 0 .$$

$[(x + 1)(x + 2)(x + 3)][1 - (x + 4)] = 0$.

$(x + 1)(x + 2)(x + 3)(-x -3) = 0$.

$-1(x + 1)(x + 2)(x + 3)^2 = 0$.

And so $x = -1$, $x = -2$, and $x = -3$ (double root).

26. $x \approx 0.618$.

*27. $P(x) = x^3 - 3x^2 + 2x - 1$

$P'(x) = 3x^2 - 6x + 2$ $P'(x) = 0$ when

$$x = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$$

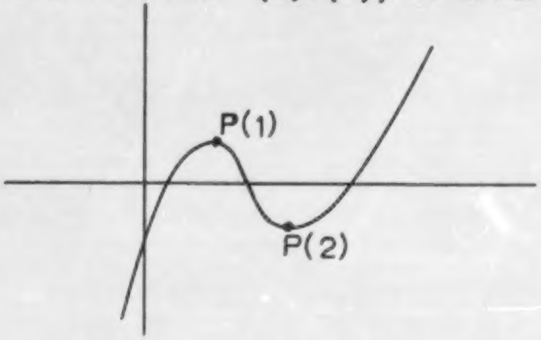
For $\frac{3 - \sqrt{3}}{3}$ we get a maximum which is below zero, and this means only one real root. $x \approx 2.324$.

*28. a) No value of $k$ since cubic polynomials will always have at least one real root. (Imaginary roots must occur in pairs.)

b) For $k > -4$ or $k < -5$.

c) For $-5 < k < -4$ as we show below.

First find $P'(x) = 6x^2 - 18x + 12$. $P'(x) = 0$ if $x = 1$ or $x = 2$. $P(1) = 5 + k$ and $P(2) = 4 + k$. $(1,P(1))$ is a minimum. $(2,P(2))$ is a maximum.



For three real roots $P(1) > 0$ and $P(2) < 0$. That is $k + 5 > 0$ and $k + 4 < 0$. Hence, finally $-5 < k < -4$.

*29.

| Maximum number of: | Positive Roots, | Negative roots. |
|---|---|---|
| a) | 2 | 1 |
| b) | 2 | 1 |
| c) | 1 | 1 |
| d) | 1 | 0 |
| e) | 0 | 1 |
| f) | 0 | 0 |

30. a) $x = 1$ (double root), $x = -1$ (triple root).

b) $x = -1$ (double root), $x = -\frac{2}{3}$ and $x = -\frac{3}{2}$.

31. $x = 2.000$

32. The conjugate of a function is the function of the conjugate.

    a) $P(2 - i) = 3 + 4i.$

    b) $P(2 + i) \cdot Q(3 + i) = (3 - 4i)(2 - i) = 2 - 11i.$

    c) $P(2 + i) \cdot Q(3 + i) = (3 + 4i)(2 - i) = 10 + 5i$

33. If complex coefficients are permitted: (a) 3; (b) 3; (c) 3.

34. If coefficients must be real: (a) 4; (b) 5; (c) 4.

35. a) $x^3 - 8x^2 + 19x - 14.$

    b) $x^3 - 5x^2 + 17x - 13.$

*36. To form a polynomial having $\sqrt{2} + \sqrt{3}$ as a zero, we set $x - (\sqrt{2} + \sqrt{3}) = 0$ . Now we have to rationalize the coefficients.

$x - \sqrt{2} = \sqrt{3}$ and squaring we get

$x^2 - 2\sqrt{2x} + 2 = 3.$

$x^2 - 1 = 2\sqrt{2}\,x$ and squaring again, we get

$x^4 - 2x^2 + 1 = 8x^2.$

$x^4 - 10x^2 + 1 = 0.$

If $\sqrt{2} + \sqrt{3}$ is rational, it is a rational zero of $P(x)$. But if $\frac{p}{q}$ (in lowest terms) is a rational zero of $P(x)$, then by the rational zeros theorem $p$ must divide 1 and $q$ must divide 1. Hence $\frac{p}{q} = + 1$, or $\frac{p}{q} = -1$. But $\sqrt{2} + \sqrt{3} \neq 1$ and $\sqrt{2} + \sqrt{3} \neq -1$. Hence $\sqrt{2} + \sqrt{3}$ is not a rational zero of $P(x)$ and therefore must be irrational.

37. a) $(2 + \sqrt{3})^n + (2 - \sqrt{3})^n =$

$$2^n + n \cdot 2^{n-1} \sqrt{3} + \frac{n(n-1)}{1 \cdot 2} \quad 2^{n-2} (3)^2 + \dots$$

$$2^n - n \cdot 2^{n-1} \sqrt{3} + \frac{n(n-1)}{1 \cdot 2} \; 2^{n-2} (\sqrt{3})^2 - \dots$$

The even numbered terms of each binomial expansion add out, so all irrationals are eliminated.

b) In this expansion, the rational terms add up to zero and the irrationals are left.

38. The four terms of the sequence can be shown (by successive differences)to be of the form $n^2 + 1$. The polynomial of least degree is $n^2 + 1$. Any polynomial which will have a term which disappears at $n = 1, 2, 3,$ or $4$ will, when added to $n^2 + 1$, give an unlimited number of polynomials. For example:

$n^2 + 1 + A(n - 1)(n - 2)(n - 3)(n - 4)(n - B)$ where A and B are any numbers or polynomials.

39. If a, b, and c are zeros of $x^3 + 7x^2 + 5$, find a polynomial with zeros $a + 2$, $b + 2$, and $c + 2$. To do this, we want a $P(x)$ having zeros $x = a - 2$. But $a = x - 2$ and we know one for which $a$ is a zero,so we substitute.

$$(x-2)^3 + 7(x-2)^2 + 5 = x^3 - 6x^2 + 12x - 8 + 7x^2 - 28x - 28 + 5$$

$$= P(x) = x^3 + x^2 - 16x + 25.$$

40. As in Eq. 39, $x = 2a + 1$ gives us $a = \frac{x-1}{2}$ and substituting in $x^3 + 7x^2 + 5$ we get $(\frac{x-1}{2})^3 + 7(\frac{x-1}{2}) + 5$. This can be multiplied by a constant without changing the zeros. So

using 8 to eliminate fractions, we get:

$$(x-1)^3 + 14(x-1)^2 + 40 = x^3 - 3x^2 + 3x - 1 + 14x^2 - 28x$$
$$+ 14 + 40 =$$

$$P(x) = x^3 + 11x^2 - 25x + 53.$$

This method can only be used when every zero undergoes exactly the same change.

## Illustrative Test Questions for Chapter 3

*1. Evaluate $\sum_{k=1}^{25} k(k+1) - \sum_{k=1}^{25} k(k-1)$.

2. Given the functions $f:x \longrightarrow x-2$, $g:x \longrightarrow x^3 - 2x^2 + 3x - 1$, and $h:x \longrightarrow x^4 - x^3 + 2x^2 - 3x + 1$, find $k(x)$ such that $[(f\cdot g) + (f\cdot h)] : x \longrightarrow k(x)$.

3. Given $f:x \longrightarrow 3x + 1$ and $g:x \longrightarrow x^2 - 2$, find $(f \circ g) - (g \circ f)$.

4. Given $f:x \longrightarrow f(x) = 2x + 1$ and $g:x \longrightarrow g(x) = x^2 - 1$. Find the solution set of the equation $(g \circ f)(x) = 0$.

5. Given $f:x \longrightarrow 3x + 5$ and $g:x \longrightarrow 2x + k$, find the value of $k$ for which $(f \circ g)(x) = (g \circ f)(x)$.

6. If $P(x) = x^3 - 6x^2 + 11x + 10$, find $P(1)$, $P(2)$, and $P(3)$.

*7. Which of the following are factors of

$$2x^{73} - 4x^{56} + x^{37} - 3x + 4?$$

a) $(x - 2)$.

b) $(x - 1)$.

c) $(x + 1)$.

d) $(x + 2)$.

8. Given $P(x) = 2x^2 - 3x - 2$, find the points on the graph of $y = P(x)$ where the graph crosses

a) the x-axis,

b) the y-axis,

c) the line $x = 7$, and

d) the line $y = 7$.

9. Find all rational zeros of $x^5 - 5x^3 + 5x^2 - 1$.

10. Find all zeros of $3x^5 + 4x^4 - 26x^3 - 42x^2 + 7x + 6$.

11. Find the root of $x^3 - 3x + 1 = 0$ between 0 and 1 correct to 2 decimal places.

12. Using Newton's method, compute the real zero of $x^5 + 5x - 1$ correct to 3 decimal places.

13. If $P(x)$ and $Q(x)$ are polynomials with real cofficients, such that $P(2 + 3i) = 1 - i$ and $Q(3 - i) = 2 - i$, evaluate $[P(2 - 3i)][Q(3 + i)]$.

14. Given the equation $(x+1)(x+2)^2(x+3) = (x+1)(x+2)(3x+5)$, tabulate the roots and their multiplicities.

15. Form an equation of minimum degree having the roots 2, -2, $3 + i$ and such that the coefficients are
    a) complex
    b) real.

16. Find quotient and remainder when $2x^3 + (2-i)x + 3$ is divided by $x + 1$.

*17. Find the solution set of the equation

$$|x|^3 - 2|x|^2 - 5|x| + 6 = 0.$$

18. Find an equation of minimum degree having rational coefficients and the roots $2 + \sqrt{3}$ and $2 + i\sqrt{3}$.

## Answers to Illustrative Test Questions for Chapter 3.

*1. $\sum_{k=1}^{25} k(k+1) - \sum_{k=1}^{25} k(k-1) = \sum_{k=1}^{25} [k(k+1) - k(k-1)]$

$= \sum_{k=1}^{25} [k^2 + k - k^2 + k] = \sum_{k=1}^{25} 2k = 2 \sum_{k=1}^{25} k.$

If students know the sum of an A.P., then it is easier to find the answer, but in any case:

$$2 \sum_{k=1}^{25} k = 650.$$

2. Since $(f \cdot g) + (f \cdot h) = f \cdot (g + h)$ we get

$(x - 2)[(x^3 - 2x^2 + 3x - 1) + (x^4 - x^3 + 2x^2 - 3x + 1)]$

$(x - 2)[x^4] = x^5 - 2x^4$

$k(x) = x^5 - 2x^4.$

3. $[(f \circ g) - (g \circ f)](x) = [3(x^2 - 2) + 1] = [(3x + 1)^2 - 2]$

$= 3x^2 - 6 + 1 - [9x^2 + 6x + 1 - 2] = 3x^2 - 5 - 9x^2 - 6x + 1$

$= -6x^2 - 6x - 4.$

4. $(g \circ f)(x) = (2x + 1)^2 - 1 = 4x^2 + 4x + 1 - 1$

$= 4x^2 + 4x.$

If $(g \circ f)(x) = 0$, then $x = 0$ or $x = -1$.

$S = \{0, -1\}.$

5. $(f \circ g)(x) = 3(2x + k) + 5$

$(g \circ f)(x) = 2(3x + 5) + k.$

We want to find $k$ if $3(2x + k) + 5 = 2(3x + 5) + k$.

$6x + 3k + 5 = 6x + 10 + k$

$2k = 5$

$k = 2.5$ or $\frac{5}{2}$

6. $P(x) = x^3 - 6x^2 + 11x + 10$

| | | | | | |
|---|---|---|---|---|---|
| 1 | - 6 | + 11 | + 10 | | |
| 1 | - 5 | + 6 | + 16 | 1 | P(1) = 16 |
| 1 | - 4 | + 3 | + 16 | 2 | P(2) = 16 |
| 1 | - 3 | + 2 | + 16 | 3 | P(3) = 16 |

*7. It should be obvious that the size of the exponents make values of 2 and -2 impossible. $2(\pm 2)^{73}$ will contain over 21 digits, while $-4(\pm 2)^{56}$ will contain only 17 digits, so the first term even when opposite in sign will be 1000 times as large.

But x = 1 or x = -1 causes the polynomial to equal zero. (x - 1) and (x + 1) are factors.

8. a) $(-\frac{1}{2},0)$ and (2,0) since $2x^2 - 3x - 2 = 0$ when $x = -\frac{1}{2}$ or $x = 2$.

b) (0,-2) since P(0) = -2.

c) (7,75) since P(7) = 75.

d) $(-\frac{3}{2}, 7)$ and (3,7) since $2x^2 - 3x - 2 = 7$ when $x = -\frac{3}{2}$ or $x = 3$.

9. Only possibilities are +1 or -1.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0 | -5 | +5 | 0 | -1 | |
| 1 | 1 | -4 | +1 | 1 | 0 | 1 |
| 1 | 2 | -2 | -1 | 0 | | 1 |
| 1 | 3 | 1 | 0 | | | 1 |

There is a triple root at x = 1. The other two roots are irrational.

10. $3x^5 + 4x^4 - 26x^3 - 42x^2 + 7x + 6$

$$\begin{array}{rrrrrr|r|r|l}
3 & +4 & -26 & -42 & +7 & +6 & & & \\
3 & 7 & -19 & -61 & -54 & -48 & 1 & & \times \\
\hline
3 & 1 & -27 & -15 & +22 & -16 & -1 & & \times \\
\hline
3 & -2 & -22 & +2 & +3 & 0 & -2 & & \checkmark \\
\hline
3 & +7 & -1 & -1 & 0 & 3 & & & \checkmark \\
\hline
3 & 6 & -3 & 0 & -\frac{1}{3} & & & & \checkmark \\
\hline
\end{array}$$

$3x^2 + 6x - 3 = 3(x^2 + 2x + 1) = 0$ when

$$x = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$$

Zeros are $x = -2$, $x = 3$, $x = -\frac{1}{3}$, $x = -1 - \sqrt{2}$ and $x = -1 + \sqrt{2}$.

11. $P(x) = x^3 - 3x + 1 = 0$ Since $P(0) = 1$ and $P(1) = -1$

$P'(x) = 3x^2 - 3$ Take $x_1 = .5$

$$x_2 = \frac{-P(x)}{P'(x)} + x_1 = \frac{-(-.375)}{-2.25} + .5 = -.167 + .5$$

$$x_3 = \frac{-(.037)}{-2.667} + .333 = .014 + .333 = .347.$$

$x \approx .35.$

12. $P(x) = x^5 + 5x - 1$ $P(0) = -1$ and $P(1) = 5$.

$P'(x) = 5x^4 + 5$ By secant method try $x_1 = .2$.

$$x_2 = \frac{-P(x)}{P'(x)} + x_1 = \frac{-.00032}{5.008} + 0.200 = -.00006 + 0.200$$

And $x \approx 0.200$.

13. Since $P(\overline{x}) = \overline{P(x)}$ for polynomials with real coefficients, and $P(2 + 3i) = 1 - i$, and $Q(3 - i) = 2 - i$, then $P(2 - 3i) = 1+i$ and $Q(3+i) = 2+i$ and their product is $1 + 3i$.

14. If $(x+1)(x+2)^2(x+3) = (x+1)(x+2)(3x+5)$, then

$(x+1)(x+2)[(x+2)(x+3) - (3x + 5)] = 0$

$(x+1)(x+2)[x^2 + 5x + 6 - 3x - 5]$

$= (x+1)(x+2)(x^2 + 2x + 1) = 0$

$(x+1)^3(x+2) = 0.$

So $x = -2$ and a triple root $x = -1$.

15. a) $(x-2)(x+2)(x - 3 - i) = x^3 - (3+i)x^2 - 4x + 12 + 4i.$

b) $(x-2)(x+2)(x - 3 - i)(x - 3 + i) = (x^2 - 4)(x^2 - 6x + 10)$

$= x^4 - 6x^3 + 6x^2 + 24x - 40.$

16\.

$$\begin{array}{lllll} 2 & 0 & (2 - i) & 3 & \\ 2 & -2 & 4 - i & -1 + i & -1 \end{array}$$

$2x^3 + (2 - i)x + 3 = (x+1)(2x^2 - 2x + 4 - i) - 1 + i.$

Quotient is $2x^2 - 2x + 4 - i.$

Remainder is $-1 + i.$

*17. Only positive roots need be tested and the negative of each of these will also be in the solution set.

$$\begin{array}{rrrr|l} 1 & -2 & -5 & +6 & \\ 1 & -1 & -6 & 0 & 1 \checkmark \\ 1 & 0 & -10 & -14 & 2 \\ 1 & 1 & -2 & 0 & 3 \checkmark \end{array}$$

So Solution Set $\{1,-1,3,-3\}$.

18. $[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = x^2 - 4x + 1$

$[x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})] = \underline{x^2 - 4x + 7}$

$x^4 - 8x^3 + 24x^2 - 32x + 7.$